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Creating Multivariable Functions

Hello, welcome to the Advanced Calculus mini-lecture on how to model multivariable functions from real world examples, and how to use such functions for optimization.

We will illustrate the topic with open box examples. Open or closed boxes and cans are maybe the simplest examples. Hundreds of these boxes and cans and variations of questions like this populate Calculus textbooks.

"An open box should be constructed. The cost for the bottom is 5 dollars per square meter. The cost for the (green) left and right side is 3 dollars per square meter. The cost for the (blue) front and back side is 2 dollar per square meter. There is also a restriction—you have to use 20 dollars for the box."
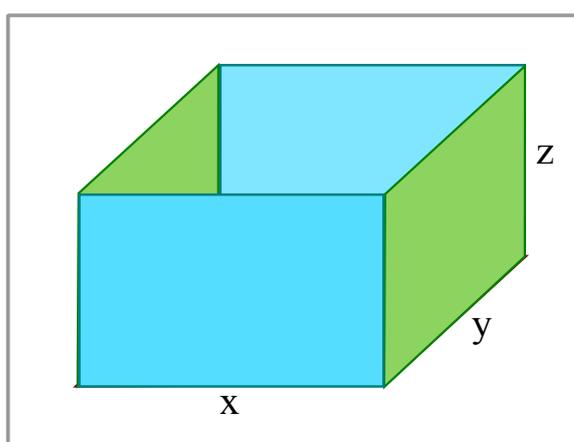

Figure 1: The open box

Of course, finding the dimensions of such an open box obeying the requirements is not too difficult, but in Calculus, the real tasks are the following:

1. Express the volume of the box in terms of the dimensions of the base.

2. Find the dimensions of such a box that maximizes the volume.

We need to label the dimensions. Let x be the width, y the depth, and z the height of such a box.

1 The Single Variable Case

So let's review this single variable Calculus I case first. Here we have an additional restriction, for instance of a square base, $x = y$. We want to express the volume of the open box in terms of the length x of the square base.

The variable we want to express, which is also the quantity we want to minimize or maximize later, we call the **target variable**. It's present value is displayed on a display. The single variable by which we want to express the target variable we call the **choice variable**. We change it using a slider control. Playing with this

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slider control is really all we can do. But indirectly, this will also influence our target variable, which we intently watch on the display. What we need to find out is how our changes of the choice variable influence the target variable on the display—this is the function we want to find in the first step.

Other than the choice and target variable, there may also be other variables. In our case we have choice variable x, the length of the square base, the target variable V, the volume of the box, and the other variable z, the height of the box.

The variables are related by some equations. The first one is the well-known volume formula for boxes:

$$V = x^2 z$$

Since we know that the total cost of the box is the sum of the cost $5x^2$ for the square base and the cost $3 \cdot 2 \cdot xz$ for the left and right sides, and the cost $2 \cdot 2 \cdot xz$ for the front and back side, and since we know that this total cost should be equal to 20, we get

$$20 = 5x^2 + 3 \cdot 2 \cdot xz + 2 \cdot 2 \cdot xz = 5x^2 + 10xz.$$

That's it. We have three variables but only two equations. This differs from Algebra word problems, where usually the number of equations equals the number of variables. But for Calculus having fewer equations is typical.

Next we have to get rid of the other variable. In our example we solve the second equation for z and get

$$z = \frac{20 - 5x^2}{10x} = \frac{4 - x^2}{2x}.$$

Then we substitute the z in the first equation by this expression and get

$$V(x) = x^2 \frac{4 - x^2}{2x} = 2x - \frac{1}{2}x^3.$$

Of course, finding the global maximum of this function is not too difficult, but let's rather move to the multivariable Calculus setting:

2   The Multivariable Case with rectangular base

In Multivariable Calculus we drop the square base requirement. Instead of one choice variable we now have two of them, width x and depth y. We have two slider controls in our advanced cockpit. When we decide about these values, the height z of the box is determined by the cost restriction. The target variable is still the volume, and we have the other variable z.

What about the equations? The volume of a box is $V = xyz$. And then there is the cost equation:

- The base rectangle has dimensions x and y, an area of xy and a cost of $5xy$.

- The left and right side both have dimensions y and z. Therefore the total area of these two sides is $2yz$ and the total cost is $3 \cdot 2yz = 6yz$.

- The front and back sides both have dimensions x and z. Therefore the total area of these two sides equals $2xz$ and the total cost equals $2 \cdot 2xz = 4xz$.

Therefore the total cost of such an open box equals $5xy + 6yz + 4xz$. And we know that this should be equal to 20, thus our second equation is

$$5xy + 6yz + 4xz = 20.$$

Note that we have four variables and only two equations. The gap between number of variables and number of equations is 2 since we have now two choice variables.

Again the other variable z we don't want to use—this variable has to go! We use our first equation $5xy + 6yz + 4xz = 20$ to eliminate z—we solve it for z and get $6yz + 4xz = 20 - 5xy$ or $(6y + 4x)z = 20 - 5xy$ or

$$z = \frac{20 - 5xy}{6y + 4x}.$$

Then we plug this into our volume equation and get

$$V = xy\frac{20 - 5xy}{6y + 4x}.$$

and our first task to express the volume — the target variable — in terms of the dimensions x and y of its base — the choice variables — is done. The target variable V is a function

$$f(x, y) = xy\frac{20 - 5xy}{6y + 4x} = \frac{20xy - 5x^2y^2}{6y + 4x}$$

of the two choice variables

In the second part of this mini lecture we will see how to find the absolute maximum of this multivariable function, but let's now rather discuss another approach:

3   The Multivariable Case with Constraint Approach

Why shouldn't all three dimensions x, y and z be choice variables? Of course, we cannot choose all three of them independently. We may have three slider controls in our cockpit, but we still have only two hands. If we work on two of them, magically the third one is moved by this. Still, a formulation like this, with a function of the target variable in terms of the choice variables and one (or more additional constraints between the choice variables is also possible, and very well-suited for Constrained Optimization using Lagrange Multipliers.

Thus our task is to maximize $V(x, y, z) = xyz$ given the constraint $5xy + 6yz + 4xz = 20$. Among all points (x, y, z) in three-dimensional space we can select those obeying this equation.

The surface of the restriction is shown below in the graph, together with one level surface of the volume function $V(x, y, z) = xyz$ we want to optimize. We look only at positive values of x, y, and z, between 0 and 2. The two surfaces touch only at one point, but still sit close together like different layers of an onion. The volume level surface is marked by the red line. Note that it is totally symmetric in x, y, and z. In contrast to this, the restriction surface is not symmetric—the behavior in the three corners of the cube which it approaches is quite different.
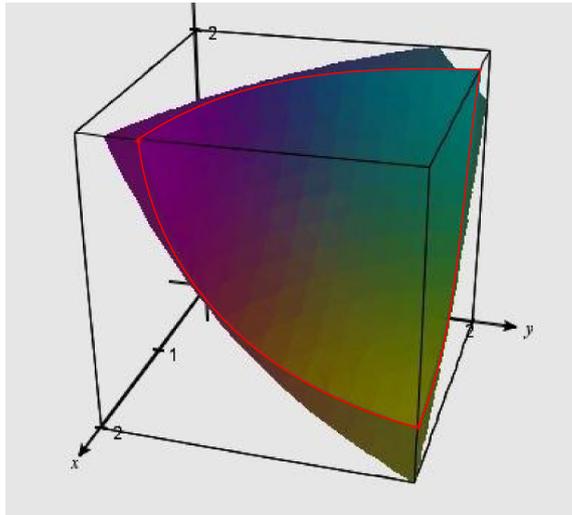
Figure 2: Our situation

All right, that's it so far. Please study absolute optimazation, and also Constrained Optimization using Lagrange multipliers, and if you have done so, you are welcome to return to the second part of this mini lecture to discuss how our concrete example is solved.