Erich Prisner

Creating Multivariable Functions II

Hello, welcome to the the second part of the Advanced Calculus mini-lecture on how to model multivariable functions from real world examples, and how to use such functions for optimization.

I hope you covered absolute optimization of multivariable functions, and optimization with constraints using Lagarange multipliers in the meantime. These two techniques are needed.

Let me briefly review what we discussed in the first part of the mini lecture. We used the following open box example:

"'An open box should be constructed. The cost for the bottom is 5 dollars per square meter. The cost for the left and right side is 3 dollars per square meter. The cost for the front and back side is 2 dollars per square meter. There is also a restriction—you have to use 20 dollars for the box."'

The tasks were

1. Express the volume of the box in terms of the dimensions of the base.

2. Find the dimensions of such a box that maximizes the volume.

Task 1 was finished in part 1, task 2 we will discuss here.

Throughout we use x for the width, y for the depth, and z for the height of such a box.

Maybe you should guess the solution here. Maybe not the values, but the relation between x, y, and z. Should all three values be about the same, in other words, should the box be a cube? And if not, which one of the three values should be largest?

1 Global Optimization

In the first part, we found that the target variable volume depends on the two choice variables x and y as follows:

$$f(x,y) = xy\frac{20-5xy}{6y+4x} = \frac{20xy-5x^2y^2}{6y+4x}.$$

Below are the level curves of the function. From them, we can already see that the maximum is around $x \approx 1.4$ and $y \approx 0.9$.

How can we find the values x and y that maximize this function f without the graph, or how can we get them precisely? Well, we need the partial derivatives: Using quotient rule we get the two partial derivatives

$$\frac{\partial f}{\partial x} = \frac{(6y+4x)(20y-10xy^2)-4(20xy-5x^2y^2)}{(6y+4x)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(6y+4x)(20x-10x^2y)-6(20xy-5x^2y^2)}{(6y+4x)^2}$$

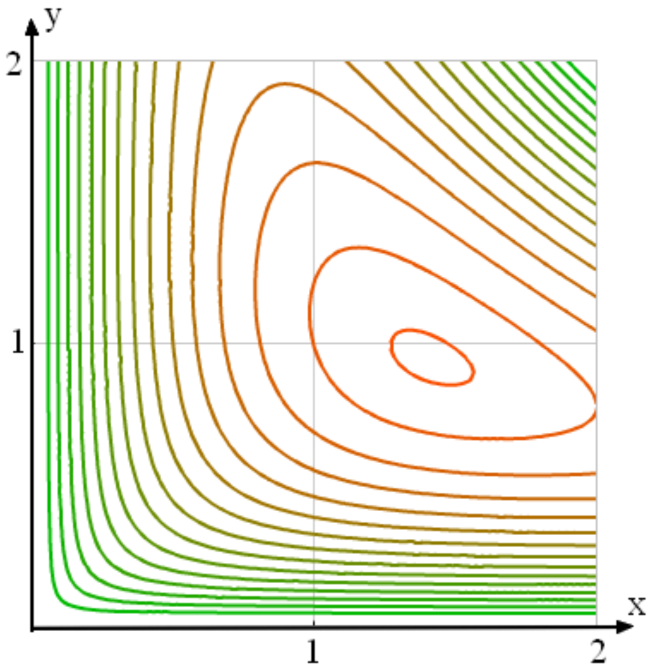

Figure 1: Level curves of the function we want to maximize

Now we need a little algebra—simplifying algebraic expressions, factoring, and expanding. The numerator of the first expression has a common factor of $10y$ which we can pull out.

$$\frac{\partial f}{\partial x} = 10y\frac{(6y+4x)(2-xy)-2x(4-xy)}{(6y+4x)^2}$$

In the same way, the numerator of the second expression has a common factor of $10x$.

$$\frac{\partial f}{\partial y} = 10x\frac{(6y+4x)(2-xy)-3y(4-xy)}{(6y+4x)^2}$$

We expand the expressions—FOIL, or use distributive law. Then in each case two of the terms vanish and we can simplify as follows:

$$\frac{\partial f}{\partial x} = 10y\frac{12y-6xy^2+8x-4x^2y-8x+2x^2y}{(6y+4x)^2} = 10y\frac{12y-6xy^2-4x^2y+2x^2y}{(6y+4x)^2}$$

$$\frac{\partial f}{\partial y} = 10x\frac{12y-6xy^2+8x-4x^2y-12y+3xy^2}{(6y+4x)^2} = 10x\frac{-6xy^2+8x-4x^2y+3xy^2}{(6y+4x)^2}$$

Then we can factor a little more, having a common factor of $2y$ in the four terms of the first expression and a common factor of x in the four terms of the second expression:

$$\frac{\partial f}{\partial x} = 20y^2\frac{6-3xy-x^2}{(6y+4x)^2}$$

$$\frac{\partial f}{\partial y} = 10x^2\frac{8-4xy-3y^2}{(6y+4x)^2}$$

Now the critical points are where both partial derivatives are equal to 0. That implies

$$\begin{aligned} y^2(6 - 3xy - x^2) &= 0 \\ x^2(8 - 4xy - 3y^2) &= 0 \end{aligned}$$

Obviously $x = 0$ or $y = 0$ does not lead to maximum volume, so we get

$$\begin{aligned} 6 - 3xy - x^2 &= 0 \\ 8 - 4xy - 3y^2 &= 0 \end{aligned}$$

The rest is algebra. Two (nonlinear) equations with two variables, substitution method, clearing the fractions of a rational equation, resulting in a quartic equation, that is in fact a hidden quadratic one, and can be solved using quadratic equation. Almost all of it basic College Algebra stuff. Please check back there if you feel insecure here.

We solve the first equation for y and get $y = \frac{6-x^2}{3x}$. When we plug this into the second equation we get

$$8 - 4x\frac{6 - x^2}{3x} - 3(\frac{6 - x^2}{3x})^2 = 0$$

or

$$8 - 8 + \frac{4}{3}x^2 - \frac{12}{x^2} + 4 - \frac{1}{3}x^2 = 0$$

or

$$x^2 + 4 - \frac{12}{x^2} = 0$$

. We multiply by x^2 and get the following quartic equation

$$x^4 + 4x^2 - 12 = 0,$$

which however, is a hidden quadratic one. We use the substitution $u = x^2$ and use quadratic formula for the resulting quadratic equation in u

$$u^2 + 4u - 12 = 0$$

and get $u = -6, 2$ or $x^2 = -6, 2$. $x^2 = -6$ has no real solution, but $x^2 = 2$ has one positive real solution $x = \sqrt{2} \approx 1.414$. Then our substitution equation $y = \frac{6-x^2}{3x}$ used above implies $y = \frac{4}{3\sqrt{2}} = \frac{2}{3}\sqrt{2} \approx 0.943$. Moreover

$$f(\sqrt{2}, \frac{2}{3}\sqrt{2}) = \frac{10}{9}\sqrt{2} \approx 1.571$$

is indeed the maximum possible volume we can get.

2 Optimization with Constraint, using Lagrange Multiplier

In the second approach we discussed in the first part we formulated the problem as

- Maximize $V(x, y, z) = xyz$
- given the constraint $5xy + 6yz + 4xz = 20$.

Let's again show the surface of the restriction, together with one level surface of the volume function $V(x, y, z) = xyz$ (with red intersection with the sides of the cube) we want to optimize, for x, y, and z between 0 and 2. The two surfaces touch only at one point, but still sit close together like different layers of an onion.

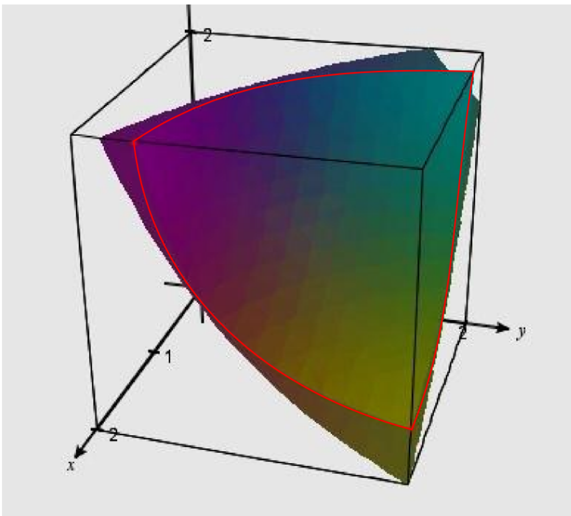

Figure 2: Our situation

We know that in the optimum, the gradient of the function $g(x, y, z) = 5xy + 6yz + 4xz$. which has our restriction equation as one of its level surfaces, and the gradient of the function V must be parallel. These gradients are as follows:

$$\nabla g = (5y + 4z, 5x + 6z, 6y + 4x)$$
$$\nabla V = (yz, xz, xy)$$

Therefore, the gradient being parallel, $\nabla V = \lambda \nabla g$ yields three equations—one for each component. The fourth equation is the restriction $g(x, y, z) = 5xy+6yz+4xz = 20$. We have a system of four equations with four variables x, y, z, and λ:

$$\begin{aligned} yz &= \lambda(5y + 4z) && (1) \\ xz &= \lambda(5x + 6z) && (2) \\ xy &= \lambda(6y + 4x) && (3) \\ 5xy + 6yz + 4xz &= 20 && (4) \end{aligned}$$

We may use substitution method. First we solve the first equation for λ as $\lambda = \frac{yz}{5y+4z}$ and plug this into the other equations. We get

$$\begin{aligned} xz &= \frac{yz}{5y + 4z}(5x + 6z) \\ xy &= \frac{yz}{5y + 4z}(6y + 4x) \\ 5xy + 6yz + 4xz &= 20 \end{aligned}$$

The first equation is equivalent to $5xyz + 4xz^2 = 5xyz + 6yz^2$ or $4xz^2 = 6yz^2$ or $4x = 6y$, $y = \frac{2}{3}x$ under the natural assumption $z \neq 0$. The second equation is equivalent to $5xy^2 + 4xyz = 6y^2z + 4xyz$ or $5xy^2 = 6y^2z$ or $5x = 6z$, $z = \frac{5}{6}x$ under the natural assumption $y \neq 0$. Plugging this into the last equation yields

$$5x\frac{2}{3}x + 6\frac{2}{3}x\frac{5}{6}x + 4x\frac{5}{6}x = 20$$
$$\frac{10}{3}x^2 + \frac{10}{3}x^2 + \frac{20}{6}x^2 = 20$$

$10x^2 = 20$, $x^2 = 2, x = \sqrt{2}$.

We get $x = \sqrt{2} \approx 1.414$, $y = \frac{2}{3}\sqrt{2} \approx 0.943$, $z = \frac{5}{6}\sqrt{2} \approx 1.179$, and for the volume we get $V = xyz = \sqrt{2} \cdot \frac{2}{3}\sqrt{2} \cdot \frac{5}{6}\sqrt{2} = \frac{10}{9}\sqrt{2}$. Everything as in the previous approach, of course.

The width x is largest, followed by height z and depth y. Isn't the reason obvious? The bottom is at 5 dollars per square meter the most expensive part, but since we have two left and right parts at 3 dollars, and two front and back part for 2 dollars, we can say that left and right (with y and z involved)) cost 6 dollars, bottom (with x and y involved) 5 dollars, and front and back (with x and z involved) only 4 dollars. Thus x is involved in middle and cheap, z in cheap and expensive, and y in middle and expensive, for this reason one should not use much y but a lot of x.

Interestingly the ratio of $x : z : y$ is $6 : 5 : 4$. Note that the costs were \$3 twice (left and right), \$5 for the bottom, and \$2 twice again (front and back), so the numbers 6, 5, 4 are echoed in the prices. I leave it to you to find out whether this is always the case in box problems like this.