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Creating Functions for Optimization

Hello, welcome to the Calculus mini-lecture on how to create functions from real world examples, and how to use such functions for optimization.

Look at the following word problem: "An open box with a square base should be constructed, having a total surface area of 150 square units and a volume of 100 cubic units. Find the dimensions of the box."

Now compare it to this word problem that might occur in a Calculus course: "An open box with a square base should be constructed, having a total surface area of 150 square units. Find the dimensions of the box maximizing the volume."

Just a few changes, but they make a lot of a difference. You may notice that fewer information is given in the Calculus version. The volume restriction is missing. Instead the words "maximizing" or "minimizing" will appear.

In the Algebra version, we have two variables, the length x of the base, and the height h. We know that the total surface area is the sum of the base area x^2 and four times the area of the sides xh, and this sum is equal to 150, so we have the first equation

$$x^2 + 4xh = 150.$$

Moreover we know that the volume x^2h equals 100, so we have the second equation

$$x^2h = 100.$$

Two equations with two variables—that is about all we need to do our algebra method, which would be substitution method.

In the Calculus version, the volume is not known, therefore it must be a variable. Overall we have three variables: x, h, and the volume V. But we still only have two equations:

$$x^2 + 4xh = 150.$$

$$x^2h = V.$$

Three variables but only two equations. We will not be able to uniquely solve this system. But that is also not required. We have less restrictions, we have more freedom, we are free to make a choice. Only if you have a choice, you can maximize or minimize something.

Obviously we are most interested in the volume V—we need to make it as large as possible, but of course we cannot just decide that it should equal, say, 1 billion. The volume we cannot decide, but just observe. We call such a variable the **target variable**. What we **can** decide is x, or maybe also h, but not both, since they are interrelated. This is quite typical for such situations in Calculus I, that we can choose just one variable. We call this the **choice variable**. Imagine that we can change it using a slider control. Playing with this slider control is really all we can do. But changing x will influence h, and indirectly this will also influence our target variable, which we intently watch on the display. What we need to find out is how our changes of the choice variable influence the target variable on the display.

In other words, after defining the target variable and the choice variable, our first task is to find out how the target variable depends on the choice variable, to express the target variable in terms of the choice variable. In our case, to express volume V in terms of base length x.

For this we have to get rid of the other variable h. In our example we solve the first equation for h and get

$$h = \frac{150 - x^2}{4x}.$$

Then we substitute h in the second equation by this expression and get

$$V(x) = x^2 \frac{150 - x^2}{4x} = \frac{75}{2}x - \frac{1}{4}x^3.$$

Let me just review the steps done so far briefly. You start with a description of the situation and of the task—what you want to maximize or minimize. Next you identify all your variables, all your equations, and what the choice variable and what the target variable are. Check whether you have one equation less than variables. From this system of equations you work exactly as you would work in an Algebra problem: You use substitution method, which means that repeatedly you solve one of the equations for one of the variables, and then substitute this variable in all other equations by the corresponding expression equal to it. Just make sure in the Calculus setting that the variables you eliminate are not choice or target variable, but rather any of the others. Eventually you will arrive at one equation in choice and target variable. You then have to solve it for the target variable, and, voila—here is the target variable expressed in terms of the choice variable.

Note that up to here no concept or tool from Calculus is needed. This is just plain Algebra. Only the second step, taking this function and finding out where the function achieves its maximum or minimum value, requires Calculus.

If you don't know yet how to do this, stop here, for all others I give a brief description how to finish in our example. W differentiate the function,

$$V'(x) = \frac{75}{2} - \frac{3}{4}x^2,$$

find all critical points—values x where $V'(x)$ is either not defined or equal to 0— which in our case results in $\frac{75}{2} = \frac{3}{4}x^2$, or $x^2 = 50$, or $x = \sqrt{50} \approx 7.1$. For the height we get $h = \frac{150-50}{28.4} \approx 3.5$, and for the volume $V(7.1) \approx 176.8$.