Change of Degrees, Diameter, and Size for some Graph Operators

Erich Prisner[1] eprisner@fc.edu

[1]*Franklin College, Via Ponte Tresa 29, 6924 Sorengo-Lugano, Switzerland, building on older research that was supported by the DFG (grant no. Pr324/10-1).*

Keywords: Graph operators, distance, degrees, random graphs, computational experiments

2010 MR Subject Classification: 05C76

Graph operators transform graphs into graphs. A main theme in the literature on graph operators is how parameters change under specific graph operators.

In this talk we will look on degrees, distances, diameter, and size. The reason for this selection is the the hope that there are some graph transformations that would transform "good" small interconnection networks in even better large interconnection networks. Note that very often, the transformed graphs are much larger than the input graphs. Obvioulsy, "good" interconnection networks should have small degrees, small diameter, among other properties as high connectivity. Transformations that increase degrees and diameter only slightly but increase the size of a graph a lot would be good candidates of such "bootstrap" graph operators.

For digraphs, the line digraph operator seems to be such a bootstrap operator: De Bruijn and Kautz digraphs, originally defined otherwise, are just iterated line digraphs of reflexive or irreflexive complete digraphs [2]. Recall that the line digraph $\mathcal{L}(D)$ of a digraph $D = (V, A)$ has the arc set A of D as vertex set, and there is an arc from xy to zw in $\mathcal{L}(D)$ if and only if $y = z$. Later it was discovered that the Butterfly networks are also iterated line digraphs. In [2], Fiol, Yebra and Alegre de Miquel proposed to try iterated line digraphs $\mathcal{L}^k(D)$ of other start digraphs D for their use as *directed* interconnection networks. They presented a small example that perfomed quite well, but to my knowledge, this line of research has never been pursued further. But what about operators transforming *undirected* graphs into *undirected graphs*?

Line graph and clique graph are maybe the most well-known graph operators. The *line graph* $L(G)$ of a graph is the intersecton graph of all edges of G, meaning that the vertices of $L(G)$ are the edges of G, and two such vertices are adjacent in $L(G)$ whenever the corresponding edges in G have a common vertex. The *clique graph*

$K(G)$ of a graph is the intersection graph of all maximal cliques of G.

The line graph and clique graph change the diameter only by at most 1, up or down, see [4] and [5]. Actually $\text{diam}(\Phi(G)) \leq \text{diam}(G) + 1$ for every intersection graph operator —operators where $\Phi(G)$ is defined as the intersection graph of certain subgraphs of G— provided every edge is covered by at least one of these subgraphs. But even for some other graph operators the diameter increases only slowly. Consider for instance the *k-sequence graph* $\mathcal{S}_k(G)$ of a graph G. It has all length-k walks (i.e. sequences $x_0, x_1, \ldots, x_k$ of vertices, with each pair x_i, x_{i+1} adjacent) of G as vertices. Two such walks $x_0, x_1, \ldots, x_k$ and $y_0, y_1, \ldots, y_k$ are adjacent in $\mathcal{S}_k(G)$ if $x_i = y_{i+1}$ for all $0 \leq i \leq k-1$, or $y_i = x_{i+1}$ for all $0 \leq i \leq k-1$, or $x_i = y_{k-1-i}$ for all $0 \leq i \leq k-1$, or $y_i = x_{k-1-i}$ for all $0 \leq i \leq k-1$ (compare [7]). Fiol, Yebra, and Fabrega showed that $\text{diam}(\mathcal{S}_k(G)) \leq \text{diam}(G) + k$ for every positive integer k.

For the maybe more well-known *k-path graph* $\mathcal{P}_k(G)$, defined as the subgraph of $\mathcal{S}_k(G)$ induced by all vertices corresponding to walks that are paths ([6]), Gerloff showed that $\text{diam}(\mathcal{P}_k(G)) \leq \text{diam}(G) + k$ provided G is connected, has diameter at least k and girth at least $k + 1$ [3], but unfortunately examples show that the requirements cannot be dropped.

Thus, we will identify a few more graph operators Φ for which there is not necessarily a bound of the form $\text{diam}(\Phi(G)) \leq \text{diam}(G) + k$ but where such a bound is true almost surely for random graphs $G \in \mathcal{G}(n, n^{-t})$, random graphs with n vertices and edge probability n^{-t}, as $n \to \infty$. For instance,

- almost surely $\text{diam}(\mathcal{P}_k(G)) = 2k$ for for random graphs $G \in \mathcal{G}(n, n^{-t})$, where $0 < t < \frac{1}{2}$, and $k \geq 2$.

For another example, look at the *k-line graph* $L_k(G)$ of a graph G, whose vertices are all K_ks in G, and where two such (distinct) vertices are adjacent if the corresponding K_ks differ in only one vertex. Then

- $\text{diam}(L_k(G)) \leq 2k-1$ for almost every random graph $G \in \mathcal{G}(n, n^{-t})$ with $k \geq 3$ and $0 < t < 1/(2k-2)$.

It can also be shown that although there are many graphs G where the diameter of its k-th iterated line graph is less than the upper bound, i.e. $\text{diam}(L^k(G)) < \text{diam}(G)+k$,

- for a random graph $G \in \mathcal{G}(n, n^{-t})$ with $0 < t < \frac{1}{2}$. $\text{diam}(L^k(G)) = k+2$ almost surely.

The corresponding questions on the increase of the degrees are investigated for line graphs, iterated line graphs, k-sequence graphs, k-path graphs, and a few others.

For most operators there are useful bounds for the maximum degree of $\Phi(G)$ in terms of the maximum degree of G. But again, these maximum degrees of transformed graphs of random graphs can be decribed rather precisely.

Combining the bounds obtained on diameter, maximum degree, and size, allows us to compare graph operators for guarantees for smallness of diameter and maximum degree for a given vertex number. Probably more important than this worst case analysis would be the analysis of transformed graphs of random graphs, using the random graph results mentioned above.

Since results about random graphs are true only asymptotically and may not carry much practical relevance for small graphs with only thousands of vertices, I performed computational tests on the behavior of these operators for many small random graphs. Starting either with good small interconnection networks, or with random graphs, I computed the tranformed graphs, which were mainly in the range of 100 to 10000 vertices, and compared their features. The outcomes are quite impressive, supporting the conclusion from theoretical analysis of random graphs that k-sequence graphs and another variant may perform best.

The results for graph operators are also compared, both theoretically and experimentally with underlying graphs of line digraphs of random digraphs.

References

[1] H.J. Broersma, C. Hoede, Path graphs, *J. Graph Th.* 13 (1989) 427-444.

[2] M.A. Fiol, J.L.A. Yebra, I. Alegre de Miquel, Line digraph iterations and the (d, k) digraph problem, *IEEE Trans. Computers* C-33 (1984) 400-403.

[3] K. Gerloff, Über k-Path Graphen, Diploma Thesis, Hamburg 1998.

[4] B. Hedman, Clique graphs of time graphs, *J. Combin. Th. B* 37 (1984) 270-278.

[5] B. Hedman, Diameters of iterated clique graphs, *Hadronic J.* 9 (1986) 273-276.

[6] H.J. Broersma, C. Hoede, Path graphs, *J. Graph Th.* 13 (1989) 427-444.

[7] M.A. Fiol, J.L.A. Yebra, J. Fabrega, Sequence graphs and interconnection networks, *Ars Comb.* 16-A (1983) 7-14.