05C62: A Journey through Intersection Graph County

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Abstract

The theory of intersection graphs will soon have, together with others, an own matematics subject classification number 05C62. This promotion may be mainly due to the fact that intersection graphs have nice applications—some of them even originated in such applications. Although a large amount of work has been done for several special classes of intersection graphs, there are still only very few basic concepts and ideas which work for general intersection graphs, and even these ideas are not as widely known as they maybe should. So let me introduce 05C62, at least the intersection graph part, in this paper to you. Whereas most surveys concentrate on geometric intersection graphs, my emphasis will be on so-called 'discrete' models. The discussion of geometric models and the Helly property will hopefully also make clear why the famous models, as interval graphs, are relatively easy to deal with, and where the difficulties for the others lie. A number of variants or generalizations, like ℓ-intersection, intersection multigraphs, intersection bigraphs and digraphs, and connection graphs, will also be treated.

Example 1 *Students visit courses. Every student has a list of courses she or he visits, and every course has a list of attendant students. However, these lists are not available to the public. Now the courses have to be scheduled in such a way that no two courses with common students have overlapping meeting times. Somehow the administration samples the information which courses have common students. This information is visualized as the 'conflict graph', where the courses are the vertices, and two such vertices are joined by an edge iff there is some time conflict, i.e. if the courses have common students.*

We call such a graph an intersection graph. Even this simple example may convince you of two facts: Firstly, such intersection graphs appear at many occasions, as we shall show by several examples throughout the paper. Secondly, the intersection graph does not contain *all* the information of the situation—in the above example, in general the course lists of the students and the student lists of the courses can not be derived from our information. However, in many situations we have to deal with this piece of information, and make the best of it

Families of sets $(S_x/x \in V)$ are called *hypergraphs*. The sets S_x are called the *hyperedges*, whereas the elements of $A = \bigcup_{x \in V} S_x$ are the *points* or *vertices* of the hypergraph. The *intersection graph* $\Omega(H)$ of a hypergraph $H = (A, (S_x/x \in V))$ has V as vertex set, and two distinct vertices x, y are joined by an edge whenever $S_x \cap S_y \neq \emptyset$.

The intention of this paper is to lead students and nonexperts through various topics of the fascinating and useful research area of intersection graphs. We shall visit most parts of the area, but we do not try to achieve completeness. Although basics of geometric intersection models are also treated, the emphasis is on so-called discrete models of intersection, and generalizations as ℓ-intersection, and intersection multigraphs, bigraphs, and digraphs.

It seems to me that the area is somewhat underdeveloped, in the sense that there are only very few important general ideas working for several intersection models, which, since the topic

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as a whole is not so widely known, are rediscovered again and again. My main priority is to cover these principles, even at the cost of neglecting other elegant and interesting results that hold only for certain special models. The paper should also serve as an invitation to work into intersection graph theory.

1 More motivating examples

Example 2 *Five persons enter some train at different doors in Santiago, and leave in Concepcion at different doors. During the journey, some of them move along the train. The train is narrow enough that people passing each other must see themselves. During the journey, A saw B and E, B saw A and C, C saw B and D, D saw C and E, and E saw D and A. Is that possible?*

Example 3 *The straight lines in Figure 1 are airplane routes. They have to be assigned different flight altitudes such that, for security reasons, intersecting lines get different altitudes. How many different altitudes are necessary?*

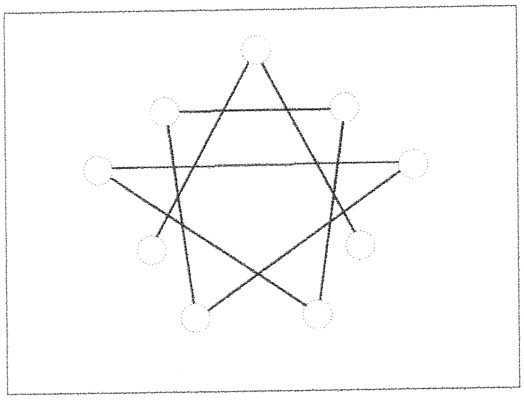

Figure 1

Example 4 *A new service at a movie theatre is to ask every viewer the range of brightness and the range of sound volume he or she would accept. According to these votes the projectionist chooses that adjustment that satisfies most people. How does he find it?*

Example 2 is a recognition problem, i.e. the question whether or not some graph is the intersection graph of sets of a certain shape. In examples 3 and 4 we have an optimization problem over some family of sets. Cruical is that the parameter only depends on which of the sets intersects. Then we may and do attack the problem by viewing the intersection graph. The parameter which has to be optimized usually is a well-known graph parameter. For instance, in example 3 we are trying to 'color' the vertices of the intersection graph by altitutes in such a way that adjacent vertices get different colors—this is just the well-known graph coloring problem. In Example 4 it turns out (see Subsection 3.4) that we are interested in a maximum clique of the intersection graph, i.e. a maximum set of pairwise adjacent vertices.

So what is the advantage of the reformulation as a graph problem? Firstly we delete unnecessary information and may concentrate on the essential. Moreover, even though the problems may and will be NP-complete in general, there is a huge amount of knowledge about such graph optimization problems, and one could try heuristics, approximation algorithms, and so on.

2 Evergreens

2.1 Line graphs

Example 5 *Assume that every student in Example 1 visits exactly 2 courses, but nobody visits* *the same two. Assume students know each other if and only if they attend some common course. All students are asked who knows whom. Is it possible to reconstruct all atte ndance lists?*

Graphs occuring in this way are called *line graphs.* More precisely, the *line graph* $L(G)$ of a graph $G = (V, E)$ has the edge set E as vertex set, and two such former edges are adjacent in $L(G)$ whenever they share a common vertex in G. Thu s it is the intersection graph of G, if edges are viewed as 2-element subsets of the vertex set.

Line graphs appeared implicitely in a 1932 paper of H. WHITNEY [W32] and were explicitely introduced by J. KRAUSZ in 1943 [K43].

¿From the beginning, the question whether and how line graphs G can be recognized, and how so-called *roots* H—graphs for which $L(H) = G$ holds— may look like for line graphs G was central in the investigation of line graphs. WHITNEY showed that every connected graph except K_3 has at most one root.[1] KRAUSZ gave in his paper a characterization of line graphs, which, however, didn't lead to a efficient recognition algorithm. But such algorithms follow from characterizations of line graphs in [RW65], [B67], see [L74], [R73], for instance.

Thus the question in Example 5 can be answered in the affirmative. If the knowledge graph of the students is connected, and if there are at least 4 students, then we may find out all attendence lists in polynomial time, and these lists are unique (modulo relabeling of the courses—of course we don't know the names of the courses).

2.2 Interval graphs

Example 6 *The mathematics department has a small cafeteria, so small, indeed that it is impossible to oversee anybody who is there at the same time. Anybody claims to have been there only once, but 8 meals have been eaten by 7 people, so somebody must have been the re twice. The investigator asks everybody whom he or she saw at the cafeteria, and the result is the graph in Figure 2. Who is the overeater?*[2]

An *interval graph* is an intersection graph of a family of intervals of the real line. These graphs have been introduced by G. HAJOS[3] in 1957. They got some kind of fame, however, after the molecular biologist R. BENZER tested in 1961 the hypothesis that the subelements of genes are linearly linked together in the following way: Under the mutations, he identified 32 so-called *deletions*—mutations caused by the loss of a connected DNA fragment. By recombination tests, it is possible to decide whether deletions overlap or not[4] The resulting graph was, as should be the case, an interval graph. One year later, LEKKERKERKER and BOLAND gave two 'good' characterizations of interval graphs, and nowadays several efficient recognition algorithms exist.

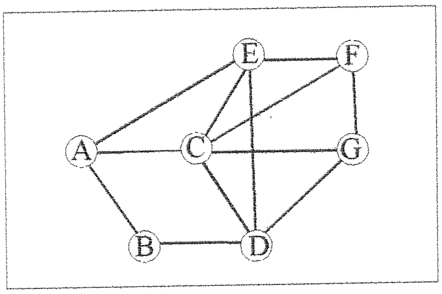

Figure 2: Who saw whom?

[1] K_3 has two roots: $K_{1,3}$ and K_3.

[2] This is a shortened version of Berge's example in [C. Berge, Les graphs d'intervalles, ...] and Golumbi c's example in [G80].

[3] G. Hajos, Über eine Art von Graphen, *Intern. Math. Nachr.* 11 (1957)

[4] to be more precise, the deletions were crossed with point mutations. A deletion which overlaps a point mutation cannot recombine with it. Thus deletions and point mutations should form the hyperedges and points of some interval hypergraph, if the hypothesis of linearity is true.

2.3 Geometric versus discrete models

Most models of the literature are variants or generalizations of these two classical examples. Either we consider intersection graphs of subsets of $\mathbf{R}^d$ with some geometric property. For the other type, we have intersection graphs of finite sets.

2.4 Line graphs and interval graphs behave well

The classes of line graphs and interval graphs are both algorithmically relatively tame: Membership in both classes can be recognized in polynomial time, furthermore many optimization problems that are NP-hard for general graphs are polynomial-time solvable when restricting the input to interval or line graphs[5] It seems that many researchers, when facing new, apparently difficult, problems, routinely try line graphs and interval graphs. This, and the prominence of interval graphs and line graphs seduced some people to take the opinion that problems for intersection graphs should always be relatively easy to tackle[6]. In my opinion, this is far from being true, and I will present proof in the remainder of the paper.

3 Intersection classes

One might think that intersection graphs are very special graphs, but this is not the case. Actually every graph $G = (V, E)$ is an intersection graph [M45]. The construction is very simple. For every vertex $x \in V$ we define S_x as the set of those edges containing x. Obviously two distinct vertices are adjacent iff they lie in some common edge, that is, iff $S_x \cap S_y \neq \emptyset$. But in the applications, usually one is interested in intersection representations of a certain type only.

In the most general situation we have some hypergraph property $\mathcal{H}$ and are interested in intersection graphs of members of $\mathcal{H}$. General intersection classes may be very difficult to handle. The class of clique graphs, introduced in Section 6, is an example of a tough class.

However, if the class $\mathcal{H}$ is closed under subhypergraphs[7], then the intersection class is closed under induced subgraphs[8]. Intersection graphs of linear 3-uniform hypergraphs, treated in Subsection 7.1, are an example of such a class.

3.1 Scheinerman classes

The simplest possible restrictions are restrictions on the sets S_x. That is, we have some set property $\mathcal{P}$ and are interested in intersection graphs of hypergraphs where all sets S_x have to obey the property $\mathcal{P}$. We will even require that the all sets obeying property $\mathcal{P}$ form a *set* which we will denote by $\mathcal{P}$ too. The class of all such finite (!) intersection graphs will be denoted by $\mathcal{G}(\mathcal{P})$.

Sometimes we do not allow multiple sets, i.e. we require the hypergraph to be simple. The class of intersection graphs of *simple* hypergraphs where all hyperedges obey property $\mathcal{P}$ is denoted by $\mathcal{G}_0(\mathcal{P})$.

Classes of the form $\mathcal{G}(\mathcal{P})$ or $\mathcal{G}_0(\mathcal{P})$ are called a *Scheinerman classes.* Mostly the two concepts do not make much difference, at least as far as recognition is concerned. Actually it can be proved that recognizing members of $\mathcal{G}(\mathcal{P})$ cannot be more difficult than recognizing members of $\mathcal{G}_0(\mathcal{P})$. It might be easier, see Section 9 for a further discussion. To explain this connection between the classes, we have to define the *reduct* $R(G)$ of a graph G as follows: Two vertices x and y are

[5]There are few exceptions for line graphs: Coloring, for instance, is still NP-complete for line graphs (since it is equivalent to edge coloring arbitrary graphs). The Hamiltonian cycle problems is also NP-complete for line graphs. For interval graphs, no obvious example is known to me

[6]Even I wrote this in one of my publications, and somehow seem to believe in it still—you wouldn't spend so much time on a topic you believe intractable.

[7]if every hypergraph obtained from a member of $\mathcal{H}$ by deleting some hyperedges is again in $\mathcal{H}$

[8]every graph we obtain from a member of the class by deleting some vertices and all incident edges lies again in the class

equivalent if they have the same closed neighborhood. This is an equivalence relation, admissible with the adjacency relation, i.e. if x, y are equivalent, and if z, v are equivalent, then x and z are adjacent if and only if y and v are. Therefore the quotient graph $R(G)$ can be defined, which has the equivalence classes as vertices, and two such classes adjacent iff any two vertices of the corresponding classes are adjacent in G.

Proposition 3.1 *A graph G lies in $\mathcal{G}(\mathcal{P})$ if and only if its reduct $R(G)$ lies in $\mathcal{G}_0(\mathcal{P})$*

Proof: Note that we can view the reduct to be a subgraph induced by a complete system W of representants of the equivalence classes.

Assume $G = \Omega((A, S_x/x \in V)))$ where all S_x obey $\mathcal{P}$. Then $R(G) = \Omega((A, S_w/w \in W)))$. If S_w and S_u were equal, for $u \neq w \in W$, they would have to be equivalent in $R(G)$, and in G also, a contradiction.

Conversely, let $R(G) = \Omega((A, S_w/w \in W)))$, with all S_w distinct and obeying $\mathcal{P}$. For every $x \in V \backslash W$ there is some $w \in W$ equivalent to x in G. We define $S_x := S_w$ and get a representation of G by $\mathcal{P}$-sets in this way. QED

Note that $R(G)$ can be computed in polynomial time.

Some examples of set properties $\mathcal{P}$ and the corresponding Scheinerman classes occuring in the literature are:

$\mathcal{P}$	$\mathcal{G}(\mathcal{P})$
intervals in $\mathbf{R}$	interval graphs
length-1 intervals in $\mathbf{R}$	unit interval graphs
graphs of continuous functions $f : [0,1] \to \mathbf{R}$	cocomparability graphs
Jordan curves in $\mathbf{R}^2$	string graphs
circular arcs of the unit circle	circular-arc graphs
chords of the unit circle	circle graphs
disks of radius 1 in $\mathbf{R}^2$	unit disk graphs
2-element sets	line graphs of multigraphs

$\mathcal{P}$	$\mathcal{G}_0(\mathcal{P})$
k-element sets	int. graphs of k-uniform hypergraphs
2-element sets	line graphs

Since in general, we can slightly change the sets of a geometric intersection representation without affecting its intersection graph, the distinction between $\mathcal{G}(\mathcal{P})$ or $\mathcal{G}_0(\mathcal{P})$ is only important for discrete intersection models.

Surprisingly, if we are ponly interested in finite graphs, we may require that $\mathcal{P}$ is countable:

Lemma 3.2 *[S85] For every Scheinerman class $\mathcal{G}(\mathcal{P})$ there is a countable subset $\mathcal{P}'$ of $\mathcal{P}$ such that $\mathcal{G}(\mathcal{P}) = \mathcal{G}(\mathcal{P})'$.*

Proof: Since for every integer n there are only finitely many graphs with n vertices, $\mathcal{G}(\mathcal{P})$ is countable, say $\{G_1, G_2, \ldots\}$. For every G_i we choose a fixed representation by $\mathcal{P}$-sets. For every such i, only finitely many $\mathcal{P}$-sets are needed, therefore the set $\mathcal{P}'$ of all those $\mathcal{P}$-sets used in this way is countable. QED

Scheinerman classes have certain properties:

(1) If G is a member of the class, then every induced subgraph must also be a member of the class. [9]

Even a stronger property holds:

[9] If $W \subseteq V$ and $G = \Omega((A, (S_x/x \in V)))$, then $G[W] = \Omega((A, (S_x/x \in W)))$.

(1′) There is some countable graph U such that the members of the class are just the finite induced subgraphs of U. [10]

The third property holds only for classes of the type $\mathcal{G}(\mathcal{P})$:

(2) If $R(G) = R(F)$ and if F lies in $\mathcal{G}(\mathcal{P})$, then G also. [11]

Interestingly, these two or three properties characterize Scheinerman classes:

Theorem 3.3 *[S85] For a graph class $\mathcal{G}$, there is some property $\mathcal{P}$ such that $\mathcal{G} = \mathcal{G}(\mathcal{P})$ (respectively $\mathcal{G} = \mathcal{G}_0(\mathcal{P})$) if and only if (1′) and (2) above (respectively (1′)) hold.*

Proof: Let $x(1), x(2), \ldots$ be the vertices of $U = (V, E)$ (of condition (1′)). We define $S_{x(1)}, S_{x(2)}, \ldots$ one after another, taking care at every step that $S_{x(i)}$ intersects all $S_{x(j)}, j < i, x(i)x(j) \in E$, but no $S_{x(j)}, j < i, x(i)x(j) \notin E$. This can be achieved at every step provided the 'private part' of every $S_{x(j)}$—those elements that do not lie in any other $S_{x(t)}$ chosen so far—remains infinite at any step.[12] QED

Since several graph classes obey these properties above, they have characterizations by intersection models. Examples are the classes of distance-hereditary graphs, or asteroidal triple free graphs, which are both Scheinerman classes. However, being far from natural, the models constructed in the proof of Theorem 3.2 are not very useful—in most cases they do not ressemble the structure of the graphs.

Problem 1 *For which of the classes of strongly chordal graphs, distance-hereditary graphs,... is there some* natural *set property $\mathcal{P}$ such that $\mathcal{G}(\mathcal{P})$ equals the class of graphs?*

An example of such a Scheinerman class, where the intersection model —which turned out to be very useful— was discovered only 16 years after the introduction of the class are the so-called chordal graphs. A graph is *chordal* if it contains no cycles of length 4 or more as induced subgraphs [G80].

Theorem 3.4 *[B74], [G74] Let A be the vertex set of some infinite binary tree T. A finite subset S of A has property $\mathcal{P}$ if it induces a subtree of T. Then $\mathcal{G}(\mathcal{P})$ equals the class of chordal graphs.*[13]

But why are the three classes mentioned Scheinerman classes? Property (1) is easy to check. For (1′), note that in these three examples the disjoint union of two graphs of the class lies again in the class. Then we can define U as the disjoint union of all (finite) graphs in the class.

3.2 Recognition and Reconstruction

In introducing line graphs, we have already mentioned two very natural problems. These problems arise for every intersection class. Let $\mathcal{H}$ be any hypergraph class, and let $\mathcal{G}(\mathcal{H})$ denote the class of intersection graphs of members of $\mathcal{H}$.

- **Recognition:** Given a graph G, decide whether $G \in \mathcal{G}(\mathcal{H})$.
- **Reconstruction:** Given a graph $G \in \mathcal{G}(\mathcal{H})$, find some or all $H \in \mathcal{H}$ such that $G = \Omega(\mathcal{H})$.

[10] Note that U is simply the intersection graph of the hypergraph $(\bigcup_{S \in \mathcal{P}} S, \mathcal{P})$. It is here that $\mathcal{P}$ being a countable set is required.

[11] This could also expressed as: If $N[x] = N[y]$ and if $G - x$ lies in $\mathcal{G}(\mathcal{P})$, then G also. For the proof, note that you simply have to add $S_x := S_y$ to the hypergraph.

[12] For instance, we might choose all $S_{x(i)}$ as subsets of $\mathbb{N} \times \mathbb{N}$ such that $S_{x(i)} = \{(i,j)/j \in \mathbb{N}\} \cup \{(j,i)/j < i, x(i)x(j) \in E\}$.

[13] This is not true for infinite graphs, compare R. Halin, On the representation of triangulated graphs in trees, *Europ. J. Comb.* 5 (1984) 23-28.

The second problem occurs quite naturally in situation where we only get the intersection information, but are interested in the original information. For instance, in a variant of Example 5, where nobody comes twice, one might be interested in who came first. Another natural example is the following:

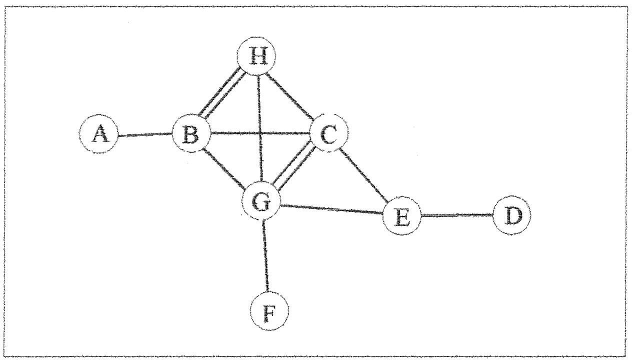

Figure 3:

Example 7 *Assume there are certain species in a phlogenetic tree. The tree is unknown and has to be reconstructed. What is known are 8 features and which of the species obey which of them. Furthermore, we may make the assumption that each of the features touches a connected part of the tree.*

We model our information by the intersection multigraph of the species-set of every feature, see Figure 3.

There are even some cases where we are interested in the model only since optimization of some parameter is doable in the model, but not in the graph alone, see Subsection 6.1.1 for an example.

4 Duality

If you look ahead to Example 8 at the beginning of Section 7, you may notice that there is some connection between this example and Example 1. There we have the intersection graph of all attendance lists of the courses, whereas in Example 8 we are considering the intersection graph of all course lists of the students. Both sets of lists obviously contain the same information. Each one is the dual of the other.

Duality is the most important, yet very natural, notion for intersection graphs. Assume there is a hypergraph $H = (A, (S_x/x \in V))$. For every $a \in A$, we define a^* to be the set of those $x \in V$ for which $a \in S_x$.

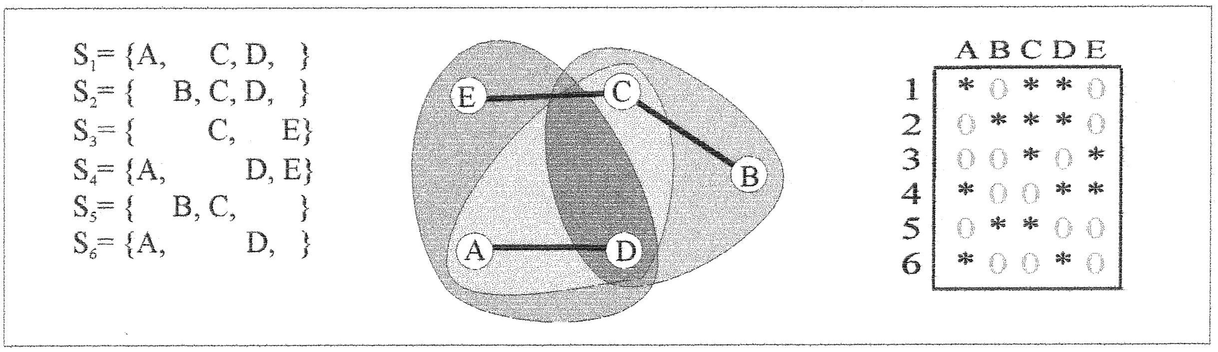

Figure 4: A hypergraph, given by its list of hyperedges, as Venn diagram, and by its incidence matrix.

The hypergraph $(V, (a^*/a \in A))$ is called the *dual hypergraph* H^* of H. A convenient way to think duals is by the *incidence matrix* of H, where the columns are labeled by the elements of

A and the rows by the elements of V, and there is a '1' (or '*', as in the figures) in row x and column a if $a \in S_x$, otherwise there is a '0'. Thus the rows 'correspond' to the hyperedges of H. See Figure 4.

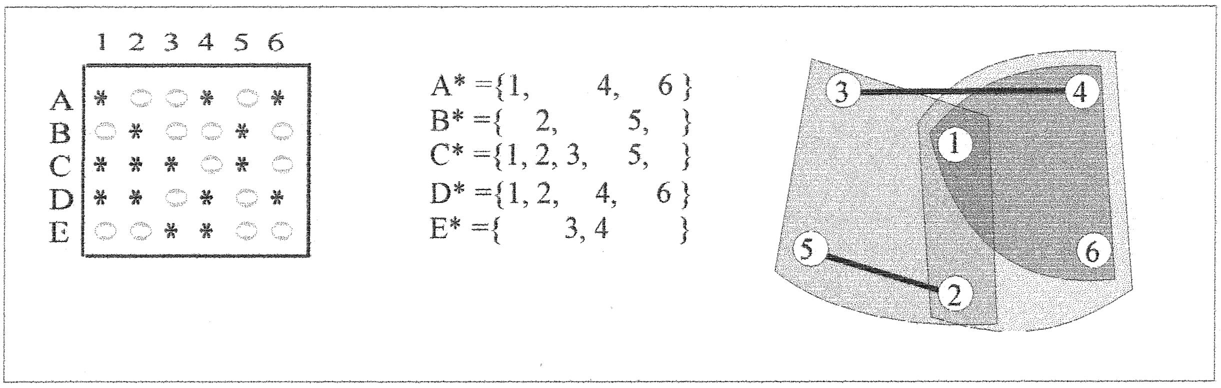

Figure 5: The dual of the hypergraph of Figure 4.

By transposing the matrix, the roles of columns and vertices are interchanged. The dual H^* is the hypergraph whose incidence matrix equals the transpose of the incidence matrix of H. See Figure 5 for an example.

By this description, it should be clear that the dual hypergraph contains the same information than the original one, in fact it can easily be retrieved from the dual since $H \simeq (H^*)^*$.

This essentially formal definition gets life by looking on intersection graphs. Let us consider the intersection graph $G = (V, E)$ of the hypergraph $H = (A, (S_x/x \in V))$. Then the dual hyperedges $a^*, a \in A$ induce certain sugraphs in G, which we call *star graphs* in the following. These star graphs have two important features:

1. Every star graph is complete, and

2. The set of all star graphs forms an *edge cover* of G, i.e. every edge of G lies in at least one star graph.

We may think of the $a \in A$ as *detectors* reporting all hyperedges covering it. Obviously two hyperedges of H intersect if and only if there is some detector reporting both. This is equivalent to the two statements above. See Figure 6 for an example how the star graphs of an intersection graph may look.

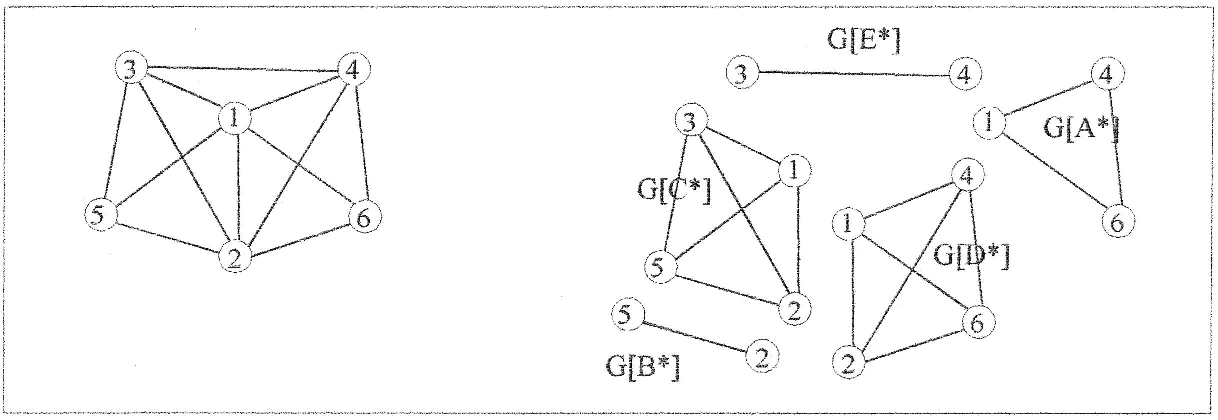

Figure 6: The intersection graph of the hypergraph of Figure 4, and its star graphs.

Intersection graphs can also be defined by duality. The *underlying graph* (sometimes called *1-skeleton*) of a hypergraph $H = (A, (S_x/x \in V))$ has the same vertex set A as H, and as edges all pairs of vertices that are covered by some hyperedge of H. Now $\Omega(H)$ is just the underlying graph of H^*—the star graphs are just the hyperedges of H^*.

Note that in geometric intersection models there may be infinitely many star graphs, however only finitely many distinct. Moreoever, since most of these geometric models are stable under slight pertuberations, we may assume in most cases that every star graph occuring occurs infinitely often.

Problem 2 *Is there any connection between intersection graphs of dual hypergraphs? Given two graphs G_1 and G_2, can you construct a hypergraph H such that $G_1 = \Omega(H)$ and $G_2 = \Omega(H^*)$?*

4.1 Characterization and Recognition via duality

For several classes of intersection graphs, recognition algorithms are available. Examples are interval graphs, chordal graphs, cocomparability graphs, line graphs. The maybe most natural approach for characterizing and recognizing graphs of some class of intersection graphs uses duality. Since it has already been used by KRAUSZ in [K43], such characterizations are usually called 'Krausz-type'. The key idea is to divide the recognition problem into two separate problems which should be solved separately, but the second after the first.

(1) The first subproblem is to find out all star graphs. Obviously, we cannot check all covers of the edge set by complete subgraphs, since their number may be exponential.

(2) If we have all star graphs, then we have H^*, and therefore also H. At first sight you may think that we are done, at least for Scheinerman classes, since all we have to check is whether the hyperedges obey property $\mathcal{P}$. But the hypergraph is only given in an abstract form, without labeling of its points. The question remains whether or not it is possible to label the points in such a way that all hyperedges obey our property $\mathcal{P}$.

If subproblems (1) and (2) can be solved efficiently, then we can recognize the corresponding intersection graphs. We shall see that subproblem (1), the problem of *identifying the star graphs*, is solvable in cases where the hypergraph obeys the so-called Helly-property, or at least does not fail too much. The second subproblem, the *layout problem*, is often easy in discrete models, but rather difficult in geometric models.

Krausz-type characterizations present characterizations of hypergraphs isomorphic to hypergraphs where all hyperedges obey property $\mathcal{P}$. Or, in the more general case of intersection graphs of hypergraphs from a class $\mathcal{H}$, they present characterizations of the class $\mathcal{H}$. Krausz-type characterizations skip subproblem 1 and only deal with subproblem 2, and even for that, the characterizations given are in some cases not polynomial-time checkable.

Theorem 4.1 *[K43] A graph G is a line graph if and only if there is some family of complete subgraphs such that every edge of G lies in exactly one and every vertex of G lies in exactly two of these subgraphs.*

Subproblem (1) becomes in most cases trivial if G is triangle-free, since then we have to take essentially all the edges as star graphs. But even for K_4-free graphs, the number of possible edge covers by complete graphs may be exponential.

4.2 The Helly property

There are two quite natural ways of covering all edges of a graph by complete subgraphs: One is by taking all edges, the other by taking all cliques. Note that cliques always are meant to be *inclusion-maximal* complete subgraphs.

That all nontrivial star graphs are edges occurs only if all intersections are 'private'— no three hyperedges have nonempty intersection. This property is rare, but can be achieved, for instance, for intersection of straight lines. Thus, in the duality approach for recognizing members of $\mathcal{G}$(straight lines), the first step is easy.

What about cliques? There are few models where all star graphs are cliques, but what about the converse? When are all cliques star graphs? Exactly if,

(H) for every set of pairwise intersecting hyperedges, the total intersection is nonempty.

This is the *Helly-property* for hypergraphs. It turns out that the restriction to intersection graphs of Helly hypergraphs is the most important reason for making things easier. We may not know all star graphs, but many of them.

Now, many natural properties $\mathcal{P}$ have the property that every system of $\mathcal{P}$-sets has the Helly-property. Examples are intervals of the real line, or more generally, d-dimensional boxes[14]. Also, subtrees of a given tree fulfill the Helly-property.

This property is named after the Austrian mathematician Eduard Helly[15]. Helly showed in 1923 that whenever every $d+1$ sets of a collection of convex sets in $\mathbf{R}^d$ have nonempty intersection, then the total intersection of these sets must be nonempty too. That coincides with our definition only for $d = 2$, but see Section 18 for applications of this so-called k-Helly property.

The dual notion of 'Helly' is 'conformal'. A hypergraph H is called *conformal* if its dual H^* has the Helly-property. Another formulation is: Every clique of the underlying graph of H should be covered by some hyperedge, that is, The clique hypergraph[16] of the underlying graph of H should be contained in H. Now it is very easy to convince you that every graph is not only an intersection graph, but even an intersection graph of some Helly-hypergraph, since every graph is the underlying graph of its own (conformal) clique hypergraph.

4.3 Example: Recognizing interval graphs

Since intervals obey the Helly-property, all cliques in an interval graph are star graphs. There are more star graphs, but we may neglect them in the remaining layout step of the recognition algorithm; we may discretize. Let $a_1^*, a_2^*, \ldots, a_k^*$ be the cliques, and define S_x' as the set of all those points a where $x \in a^*$. The question is whether or not there is some placement of these detectors $a_1, a_2, \ldots, a_k$ on the real line such that S_x' forms a consecutive part among all detectors, for every vertex x.

A *concrete interval hypergraph* has vertex set $\{1, 2, \ldots, n\}$ and all hyperedges have the form $\{i, i+1, \ldots, j\}$. An *interval hypergraph* is any hypergraph isomorphic to some concrete interval hypergraph. So we have the hypergraph, but have forgotten the vertex labels. The question above is how to recognize interval hypergraphs. Surely we have no time to test all $n!$ vertex labelings.

In matrix theory, this problem is known under the consecutive-ones problem. A 0-1 matrix has the consecutive-ones property for rows if the columns can be permuted in such a way that the ones in each row appear consecutively. A hypergraph is an interval hypergraph if and only if its incidence matrix has the consecutive-ones property for rows. It is possible, though rather involved, to recognize this property in linear time, but since this has few to do with intersection graphs, we will omit the proof.

Theorem 4.2 *[BL76] Interval hypergraphs can be recognized in linear time.*

Note that there are quite a few more characterizations leading to efficient recognition algorithms for interval graphs. In all of them, the linearity and ordering of the model is cruical.

4.4 Example: Not recognizing intersection graphs of boxes in $\mathbf{R}^2$.

As for many geometric intersection models, the first step, finding sufficiently many star graphs, is easy for intersection graphs of boxes in R^d. We only have to find the cliques. The problem is the layout step, where we have to place the cliques on the plane in a certain way. After the placement

[14] here and in the following, by *boxes* we mean generalized rectangles with sides parallel to the coordinate axes, i.e. subsets $\{(x_1, \ldots, x_d)/\forall i : a_i \leq x_i \leq b_i\}$

[15] 1884-1943.

[16] which has the same vertices than the graph, and all cliques as hyperedges

has been done, for every vertex x of G, the (points corresponding to the) cliques containing x generate a smallest axis-parallel rectangle, which we denote by S_x. These rectangles should obey:

- for every vertex x of G, all (points corresponding to) cliques in S_x must contain x, and
- if S_x and S_y intersect, then they have some (point corresponding to a) clique in common.

It is not known whether this layout problem can be solved efficiently. For higher dimensions, it must be NP-complete, since recognizing intersection graphs of d-dimensional boxes is NP-complete for $d \geq 3$ [Y82].

Problem 3 *What is the complexity of recognizing intersection graphs of 2-dimensional boxes?*

4.5 Example: Recognizing line graphs

2-uniform hypergraphs do *not* obey the Helly-property: The sets $\{a,b\}, \{b,c\}$, and $\{a,c\}$ have pairwise nonempty intersection, but empty total intersection. However, for simple hypergraph this is essentially the only such example:

Lemma 4.3 *Every clique with more or less than 3 vertices is a star graph in a line graph.*

This is the reason why the first step in the dualization-approach can be worked out, as we shall see.

We have to find out which 3-vertex cliques are star graphs and which not. We assume G connected with more than 4 vertices. Now every 3-vertex clique C which is a star graph has some vertex outside adjacent to exactly one vertex of C. On the other hand, every 3-vertex clique which is no star clique does not have such a vertex.

The final task is to add all star graphs which are not cliques. These must be 2- or 1-vertex graphs. Note that every edge lies in exactly one star clique, and every vertex in exactly two, compare Theorem 4.1. Thus we first add all edges that are not covered by clique star graphs, and after that, we add all 1-vertex star graphs necessary.

Note that the second step in the dualization-approach is trivial—it is obvious how to check a hypergraph for simplicity and 2-uniformity.

In order to be an efficient algorithm, the first step, checking all cliques, is still not performable. There are graphs, though not line graphs, with an exponential number of cliques. One has to exclude these graphs by some rough sieve in advance. One possibility is to check whether or not the graph in question has some induced $K_5 - e$—line graphs cannot have this, but graphs without induced $K_5 - e$ have few cliques, which can be generated in polynomial time.

There are quicker algorithms:

Theorem 4.4 *[R73], [L74] There is some linear-time algorithm for recognizing line graphs.*

5 Order

The most important feature of intervals of the real line is their ordering. Of two nonintersecting intervals, one is 'to the right' of the other. This relation is obviously irreflexive, asymmetric, and transitive, i.e. a partial order. This carries over to a partial order $<$ of V, and $x \neq y \in V$ are adjacent in the intersection graph if and only if they are not comparable by $<$. That means, G is the cocomparability graph (complement of the comparability graph) of $(V, <)$.

The converse is not true: Not every cocomparability graph of some poset is an interval graph. The reason is that not every poset can be represented by intervals on the real line. Those posets that can are called *interval orders*.

Proposition 5.1 *Interval graphs are exactly the cocomparability graphs of interval orders.*

But every poset can be represented by graphs of continuous functions $f : [0, 1] \to \mathbf{R}$, where we have a natural relation 'above' for functions with nonintersecting graphs. Therefore:

Theorem 5.2 *[GRU83] Cocomparability graphs are exactly the intersection graphs of graphs of continuous functions* $f : [0, 1] \to \mathbf{R}$.

Comparability graphs and cocomparability graphs can be recognized in time $O(n^{2.5})$, compare [G80] and for further information on them. There are various other geometric intersection models where such an order is apparent. Examples are permutation graphs—intersection graphs of straight line segments between two parallel lines, or trapezoid graphs—intersection graphs of trapezoids between two parallel lines. In all such classes we may use the important fact that cocomparability graphs are perfect, and many parameters can be computed efficiently for them.

Although I refer to [G80] for those classes, I would like mention one result on permutation graphs, which we will need later on. First note that any two nonintersecting lines (between two parallel lines) have a natural order. But there is also a natural order between intersecting lines: one is larger than the other if it can be obtained from the first one by tilting it in clockweise direction. Thus permutation graphs are comparability graphs as well as cocomparability graphs, but the converse is also true:

Theorem 5.3 *[DM41] A graph is a permutation graph if and only if both G and $\overline{G}$ are comparability graphs.*

Then permutation graphs can also be recognized[17] in time $O(n^{2.5})$.

Concerning Example 2, when we consider the time-space, which is here a plane, since space is 1-dimensional, we see that the resulting graphs are exactly the cocomparability graphs. But the 5-cycle of the example is no cocomparability graph, it is even not perfect. Thus the pattern is impossible.

Proposition 5.1 leads to the most famous characterization of interval graphs. Three vertices form an *asteroidal triple* if any two of them are connected by a path avoiding the third and all neighbors of the third. No cocomparability graph can have any asteroidal triple. Adding one restriction we arrive at chordal graphs:

Theorem 5.4 *[LB62] Interval graphs are exactlty the chordal graphs without asteroidal triple.*

6 Cliques

As mentioned in Subsection 4.5, line graphs have few cliques. Although not necessary for intersection classes, this property is not unfamiliar. But let us for the moment forget about intersection graphs and treat the following problem: Given is a class $\mathcal{G}$ of (finite) graphs. How can we decide whether there is some polnomial $f(n)$ such that every n-vertex graph of the class has at most $f(n)$ cliques? This is a nontrivial property, since the class of all finite graphs does not allow such a polynomial. This can be seen by the so-called *octahedron graphs* $\overline{tK_2}$, which are obtained from complete graphs with an even number of vertices by deleting a perfect matching. Every way of picking one vertex of every pair of nonadjacent vertices of $\overline{tK_2}$ gives a clique, thus $\overline{tK_2}$ has $2t$ vertices but 2^t cliques. It turns out that for classes which are closed under induced subgraphs, these graphs are essentially the only obstructions.

Theorem 6.1 *[BY89],[P95] Let the class $\mathcal{G}$ of (finite) graphs be closed under induced subgraphs. Then there is some polynomial $f(n)$ such that every n-vertex member of $\mathcal{G}$ has at most $f(n)$ cliques if and only if there is some t such that $\overline{tK_2}$ is not in $\mathcal{G}$.*

[17] This can be imoved to $O(n^2)$, compare E. Spinrad, On comparability and permutation graphs, *SIAM J. Comput.* 14 (1985) 658-670.

Moreover, we can choose $f(n) = n^{2t-2}$ in that case.

Listing all c cliques of a graph can be done in time $O(nmc)$ for every graph [TIAS77], thus if $\overline{tK_2}$ is forbidden in our class, we can list all cliques in time $O(n^{2t+1})$. If there are only few cliques, a maximum clique, or a maximum weighted clique can also be found in the same time. All we have to do is list all cliques and check.

6.1 Cliques in intersection graphs

Since Scheinerman classes[18] are closed under induced subgraphs, this result immediately applies to them. Let us just list some examples:

- $\overline{2K_2} = C_4$ is no interval graph,
- $\overline{(d+1)K_2} \notin \mathcal{G}(\text{boxes in } \mathbf{R}^d)$,
- $\overline{4K_2}$ is no line graph,

Note that the question of Example 4 can now be answered efficiently.

What happens if the number of cliques is exponential. There are intersection classes where the MAXCLIQUE problem is NP-complete, for instance intersection graphs of convex sets in the plane [KK97]. For some classes, as intersection graphs of straight lines, the complexity is open. Sometimes maximum cliques can still be computed efficiently. Examples are circular-arc graphs, but also the following example:

6.1.1 Cliques in unit-disk graphs

Every generalized octhedron $\overline{tK_2}$ is a unit-disk graph. Nevertheless it turns out that a maximum clique can be found in polynomial time provided we have the model available. This is a nice example where reconstruction could be motivated by some optimization problem. Unfortunately, it is not known whether unit-disk graphs can be reconstructed efficiently. Thus the complexity status of the maximum clique problem restricted to unit disk graphs (without a representation given) is also open.

Theorem 6.2 *[CCJ90] There is some $O(n^{4.5})$-time algorithm to find a maximum clique in a unit disk graph provided the model is given.*

Proof: First we provide a fresh view on the model. Let $G = (V, E)$ be the intersection graph of the family $(S_x/x \in V)$ of unit-disks in the plane. Let p_x be the center of S_x. Obviously S_x and S_y intersect iff the distance between p_x and p_y is at most 2. Therefore it suffices to look at these points p_x.

For $xy \in E$, let A_{xy} denote the set of all $z \in V$ for which p_z is contained in both (closed) circles around p_x and p_y with radius $\overline{p_x p_y}$. Certainly each such $z \neq x, y$ must be adjacent to both x and y in G. These sets A_{xy} are not complete, but they induce complements of bipartite graphs. This can be seen by drawing the line between p_x and p_y. Each half has diameter less than 2, therefore vertices corresponding to points in each half are pairwise adjacent.

Maximum cliques in complements of bipartite graphs, i.e. maximum independent sets in bipartite graphs, can be found in time $O(n^{2.5})$ via maximum matching in bipartite graphs[19].

Any clique C of G is contained in A_{xy} where we choose $x, y \in V(C)$ with maximum distance. Thus by finding maximum cliques in each of the n^2 graphs A_{xy}, we find a maximum clique of G. QED

Problem 4 *What about the MAXCLIQUE problem for unit-disk graphs without a representation given? Can a maximum clique at least be approximated then?*

[18] or more general, intersection graphs of members of $\mathcal{H}$, where $\mathcal{H}$ is closed under subhypergraphs,

[19] Choose a maximum matching M (see J. Edmonds, R.M. Karp, Theoretical improvements in algorithmic efficiency for network flow problems, *J. ACM* 19 (1972) 248-264), take all vertices not covered by M into S. Then it is possible to add for every edge of M one of its vertices into S and still maintain independence.

6.2 Clique multigraphs

There is another way of finding out which cliques are star graphs in line graphs. Although not really necessary there, it becomes necessary for other models presented later. We have to define the *clique multigraph* $C_M(G)$ of G. This is just the intersection multigraph of the set of all cliques of G, i.e. the cliques of G are the vertices of $C_M(G)$, and two vertices are joined by as many parallel edges as the cliques have vertices in common. See Figure 7 for an example. Now observe that for line graphs G the number of edges between every pair of adjacent vertices in $C_M(G)$ reveals whether both have the same type—star graph or not. Two star graphs have at most one common vertex. Two non star graph cliques must stem from different triangles, and therefore have also at most one common vertex. But a star graph and a non star graph have either none or two common vertices. Since $C_M(G)$ is connected for connected G, the vertices of $C_M(G)$ should partition into two sets, one of which should give all star graph cliques.

We will give three more examples where the clique multigraph plays an important role for the recognition problem in subsections 7.1, 11.1, and 16.2.2.

The first use of clique multigraphs in intersection graph theory is the following theorem, which is essentially[20] due to BERNSTEIN and GOODMAN, but which has been rediscovered several times[21], which answers the reconstruction problem completely.

Theorem 6.3 *[BG81] A chordal graph G is the intersection graph of subtrees of some tree T if and only if T is contractible to some maximum[22] spanning tree of the clique multigraph $C_M(G)$ of G.*

The proof of the Theorem uses the following characterization of chordal graphs: We say a spanning tree T of $C_M(G)$ has the *inclusion-property* if $C_0 \cap C_1 \supseteq C_0 \cap C_1 \supseteq \cdots \supseteq C_0 \cap C_t$ for every path $C_0, c_1, \ldots, C_t$ in T.

Proposition 6.4 *Let T be a maximal spanning tree of $C_M(G)$. Then G is chordal if and only if T has the inclusion-property.*

Proof: The proof is by induction on the vertex number and uses the well-known fact that every chordal graph has some so-called simplicial vertex—a vertex with complete neighborhood. The details are left to the reader. QED

Proof: of Theorem 6.3: Let T be a maximum spanning tree of $C_M(G)$. We define $T_x := T[\{C \in V(T)/x \in C\}]$ for every vertex x of G. Since T has the inclusion-property by Proposition 6.4, each T_x is connected.

If S is contractible to T, then this representation can be extended to some representation by subtrees of S in an obvious way.

Conversely, assume G is the intersection graph of subtrees $(T_x/x \in V(G))$ of some tree T. We may assume that T is a clique tree, i.e. the graphs a^* are distinct cliques of G. In other words, T is a spanning tree of $C_M(G)$. If T were not maximum, then, by Proposition 6.4, the inclusion property would be violated. Then $C_0 \cap C_i \not\supseteq C_0 \cap C_t$ for some C_i lying between C_0 and C_t on T. Then there were some vertex x occuring in C_0 and C_t, but not in C_i, a contradiction to T_x being connected. QED

Consider the version of Example 7 where we do not know how many species have both two given features, but simply whether there are such species at all. Then we only have the intersection information of the features. We may apply the theorem and get at least *some* model,

[20] they use completely different terminology

[21] Y. Shibata, On the tree representation of chordal graphs, *J. Graph Th.* 12 (1988) 421-428, but also some unpublished papers by F.V. Jensen, C.E. Carraher, and myself.

[22] the multigraph is considered as a weighted graph—every edge has its multiplicity as weight.

see Figure 17b. When we ask whether some particular tree could be the model, however, the Theorem might be of less use[23]: Actually, the following problem is NP-complete.

- **Instance:** A tree T and some weighted graph W, both with the same number of vertices.
- **Question:** Is some maximum spanning tree of W isomorphic to T?

For instance, if the tree is a path, and if all edge weights are equal, this is just the problem of finding a Hamiltonian path, if one exists.

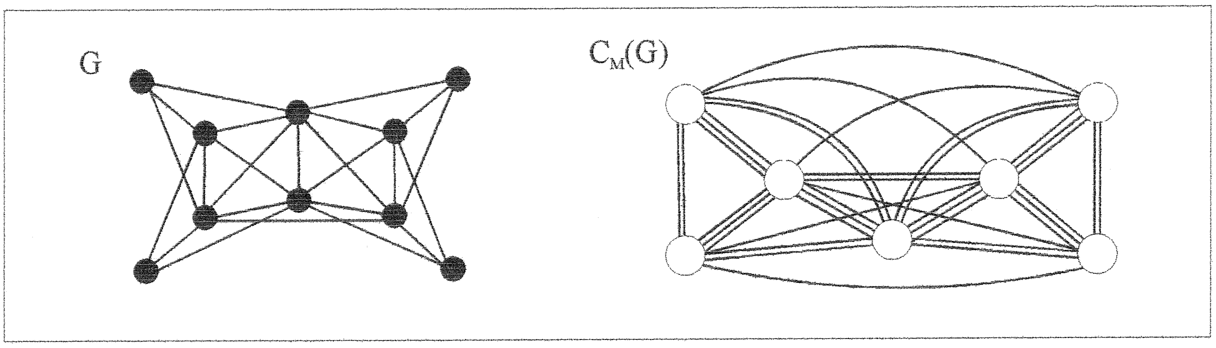

Figure 7

Problem 5 *What is the complexity of the above problem restricted to clique multigraph of chordal graphs?*

Problem 6 *Is there a good characterization of clique multigraphs of chordal graphs?*

6.3 Clique graphs

We have seen that intersection graphs of Helly-hypergraphs behave well. What about intersection graphs of conformal hypergraphs (i.e. underlying graphs of Helly-hypergraphs)? Note that we may concentrate on simple conformal hypergraphs where no hyperedge contains another, since otherwise we could modify the hypergraph slightly—by adding a new vertex for every hyperedge, contained only in that hyperedge. These hypergraphs are just the clique hypergraphs of graphs. The *clique graph* $C(G)$ of a graph G is the intersection graph of the family of all cliques of G[24] . Thus the intersection graphs of conformal hypergraphs are just the clique graphs.

Clique graphs seem to be very difficult to deal with. Other than in the Helly-case, not every graph is the intersection graph of some conformal hypergraph (i.e. clique graph) Clique graphs have an obvious dualization[25] but no good characterization. In other words, clique graphs are very interesting.

6.3.1 Clique graphs of intersection classes

Although (or maybe since) intersection graphs carry less information than intersection multigraphs, clique graphs of chordal graphs have a nice characterization:

Theorem 6.5 *[SB94] A graph G is the clique graph of a chordal graphs if and only if G has some spanning tree T such that for every $xy \in E(G)$, all vertices z on the unique x-y path in T are also adjacent to x and y.*

[23]There are, however, results in this direction which do not use Theorem 6.3, see E. Prisner, Representing triangulated graphs in stars, *Abh. Math. Sem. Univ. Hamburg* 62 (1992) 29-41, for subdivisions of $K_{1,n}$, and see Ko-Wei Lih, Ranks of chordal graphs, *Bull. Inst. Math. Acad. Sci.* 16 (1988) 357-364, for trees where there is some subpath such that every vertex of the tree has distance at most k to these path.

[24]i.e. of the clique hypergraph

[25]Roberts/Spencer ...

Actually these graphs can be recognized by some $O(n^2m)$-time greedy algorithm [SB94].

Surely, clique graphs of graphs of a Scheinerman class $\mathcal{G}(\mathcal{P})$, where $\mathcal{P}$ has the Helly property, have some nice feature: Let G be the intersection graph of certain $\mathcal{P}$-sets S_x. Then the vertices of $C(G)$ correspond to the cliques of G, which themselves correspond to certain points of $\bigcup_{x\in V} S_x$. Two such vertices of $C(G)$ are adjacent whenever the corresponding points are covered by some common S_x.

Let us give the criterion for clique graphs of intersection graphs of 2-dimensional boxes. The question is, whether the vertices of the graph H in question can be placed on the plane in such a way that for every two adjacent vertices x and y, the box generated by them covers only vertices of H adjacent to both x and y. Thus we place the vertices of H in such a way that the box generated by the placed vertices of every clique of H contains no further placed vertices of H. This is essentially the first condition of the criterion in Subsection 4.4.

Problem 7 *Can clique graphs of intersection graphs of 2-dimensional boxes be recognized in polynomial time?*

Escalante's characterization[26] of second iterated clique graphs of so-called clique-Helly graphs[27] also follow by the above observations.

7 Intersection graphs of k-uniform hypergraphs

We have seen that line graphs, i.e. intersection graphs of simple 2-uniform hypergraphs, can be recognized easily and quickly. What about intersection graphs of k-uniform hypergraphs for higher k? Let us modify Example 5 slightly:

Example 8 *Let $k \geq 2$ be any integer. Assume that every student in Example 1 visits exactly k courses. Assume students know each other if and only if they attend some common course. All students are asked who knows whom. Is it possible to reconstruct all attendance lists?*

Let us begin wit the a Krausz-type characterization:

Theorem 7.1 *A graph is the intersection graph of some (simple) k-uniform hypergraph if and only if there is some covering of its edges by complete subgraphs such that every vertex lies in at most k of them (and no two vertices lie in the same k of these subgraphs)*[28].

For $k \geq 3$, even arbitrary large cliques may be no star graphs, as can be seen for $k = 3$ in Figure 8. This is completely different to 2-uniform hypergraphs (see Lemma 4.3), which miss the Helly property only slightly, and may be the reason why the case $k \geq 3$ is much more difficult.

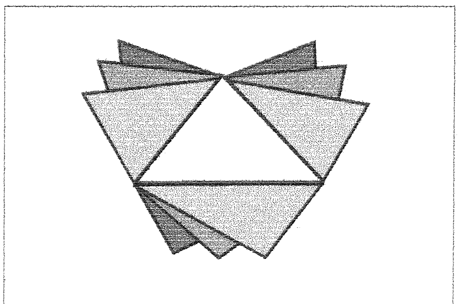

Figure 8: Pairwise intersecting 3-sets with empty total intersection.

[26] F. Escalante, Über iterierte Cliquen-Graphen, *Abh. Math. Sem. Univ. Hamburg* 39 (1973) 58-68.

[27] graphs whose clique hypergraph obeys the Helly-property

[28] but it is allowed that two vertices x, y lying in exactly t of them, where $t < k$, lie in exactly the same. We could add the one-vertex complete graphs $\{x\}$ and $\{y\}$ to the list in that case.

Theorem 7.2 *(See [PRT81] for $k \geq 3$ and [R73], [L74] for $k = 2$) Recognizing intersection graphs of k-uniform simple hypergraphs is NP-complete for every $k \geq 3$, but can be done in linear time for $k = 2$.*

See Section 9 for a proof.

Large generalized octahedra are not in the class, therefore maximum cliques can be computed in polynomial time. Therefore, maximum student cliques in Example 8 are easy to find, but note that the members of such a clique do not necessarily all attend the same course.

Theorem 7.3 *[P**] $\overline{\binom{2k-1}{k}K_2}$ is the intersection graph of some simple k-uniform hypergraph, but $\overline{(\binom{2k-1}{k}+1)K_2}$ is not.*

For every fixed integer k, intersection graphs of k-uniform hypergraphs allow an inexpensive representation of the graph by means of these sets S_x. Note that $|\bigcup_{x \in V} S_x| \leq kn$ in that case (where n and m denote the number of vertices and edges) and we may represent elements of $\bigcup_{x \in V} S_x$ by $\log(kn)$ bits. To store S_x we need $k \log(kn)$ bits for every vertex x, thus overall $kn \log(kn)$ bits, which is $O(n \log n)$ for fixed k. This is less than the space $O(n^2)$ necessary for adjacency matrices, or the space $O(m \log n)$ needed for neighborhood lists[29], but still we only need a constant number (k^2) of comparisons to check whether two vertices are adjacent.

7.1 *Linear* hypergraphs

A hypergraph is *linear* if every two hyperedges share at most one point[30]. Again a Krausz-type characterization is possible:

Theorem 7.4 *A graph is the intersection graph of some (simple)* linear *k-uniform hypergraph if and only if there are complete subgraphs such that every vertex lies in at most k of them (and no two in the same k of them) and every edge lies in exactly one of these subgraphs.*

Again the recognition problem is difficult. Again a proof will be given in Section 9.

Theorem 7.5 *[HK97] Recognizing intersection graphs of 3-uniform linear hypergraphs is NP-complete.*

Lemma 7.6 *$K_5 - e$ and $\overline{4K_2}$ are intersection graphs of linear 3-uniform hypergraphs, but $K_6 - e$ and $\overline{5K_2}$ are not.*

The following cruical lemma, which has the same type as Lemma 4.3, follows by the classification of cliques in intersection graphs of 3-uniform hypergraphs in [P**]. However, there is also a simpler proof:

Lemma 7.7 *[NRSS82], [JKL97], [HK97], [MT97] Every clique with more than $k^2 - k + 1$ vertices in the intersection graph G of a k-uniform linear simple hypergraph H is a star graph.*

Proof: Assume the hyperedges $X_1, X_2, \ldots, X_n$ with $n > k(k-1)+1$ in a linear hypergraph are pairwise intersecting. By the pigeon-hole principle, some point a of X_1 is covered by at least k of the sets $X_2, \ldots, X_n$, say $a \in X_2, \ldots, a \in X_{k+1}$. Every $X_i, k+1 < i \leq n$, not containg a has to intersect $X_1, X_2, \ldots, X_{k+1}$ in different points, this is impossible since $|X_i| = k$. Thus all X_i contain a. QED

Therefore, other than in the general case, in the linear case all large enough cliques of the intersection graph must be star cliques. Then we can achieve polynomiality for the recognition problem by restricting the possible models:

[29] note that every item of such a neighborhood list requires $\log n$ bits

[30] Note that every linear hypergraph without hyperedges of cardinality smaller than 2 is simple

Proposition 7.8 *For every $k \geq 3$, let $\mathcal{H}$ be the class of k-uniform linear hypergraphs where every point lies in at least $k^2 - k + 2$ hyperedges. Then intersection graphs of members of $\mathcal{H}$ can be recognized in polynomial time.* [31]

The assumption can be weakened to $k^2 - k + 1$ by using the clique multigraph method. The details are very similar to the proof of Theorem 7.10.

However, it is always more desirable to get a characterization under additional assumptions on the graph, not on the model. This is possible for $k = 3$: In [NRSS82] it has been shown that it is possible to test graphs of minimum degree at least 69 whether or not they are intersection graphs of linear 3-uniform hypergraphs. The bound has recently independently been lowered to 19 by a different approach by two groups [JKL97] and [MT97]. We will sketch this approach before we show how this result can be improved using clique multigraphs.

So assume G is the intersection graph of some linear 3-uniform hypergraph, and assume $\delta(G) \geq 19$. Key of the algorithm is the fact that, if we reveal all cliques with more than 7 vertices as star graphs, every vertex must lie in at least one of these already revealed star graphs (since its at most 19 neighbors should be covered by three star graphs). Before we list all cliques, we should however check whether G is really $\overline{5K_2}$-free, as it should.

Every vertex lying in one or two of these already revealed star cliques gets 2 respectively 1 as a label. Next we delete all edges in the the revealed star cliques. Every vertex that lied in three of them should be isolated now, and is also deleted. For the resulting vertex-labeled graph G' we check whether it is the intersection graph of some linear hypergraph where every S_x should have as many elements (1 or 2) as the label of x indicates. This is essentially recognition of line graphs, compare [JKL97] and [MT97], and can be done in linear time.

Note that this approach is not extendable to higher k.

Using clique multigraphs we can lower this bound further. Our goal is to reveal all star graphs with more than 5 vertices. First we need a lemma on the mutual position between star cliques and non-star cliques. Note that representations of non-star cliques must have one of the shapes of Figure 9, and these cliques must have between 3 and 7 vertices.

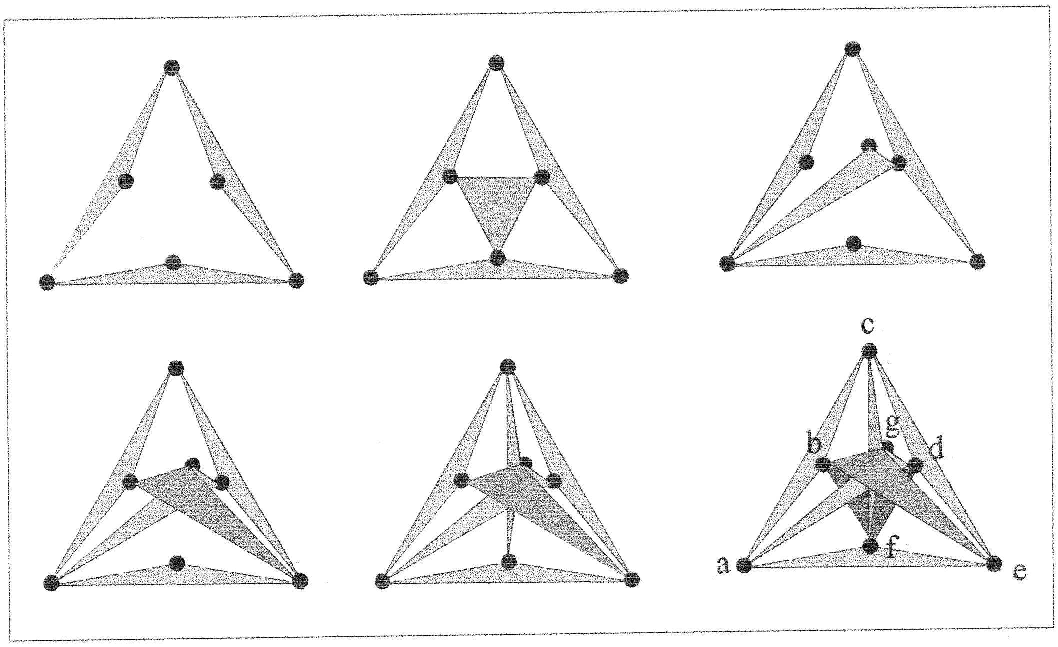

Figure 9: Non-star cliques

[31] There follows that there is some algorithm for the following problem to find H for intersection graphs $\Omega(H)$ of large random linear 3-uniform simple hypergraphs H. The algorithm succeeds in finding *some* such H in almost every case in polynomial time. In the few cases where $\delta(H) \leq 7$, either we need superpolynomial time, or we don't compute H.

Lemma 7.9 *Let C_1 and C_2 be two cliques with at least 6 vertices in the intersection graph G of some 3-uniform linear hypergraph H. If both are star graphs, or if both are hole cliques, then $|C_1 \cap C_2| \leq 1$. If one of them is a star clique and the other a hole clique, then $|C_1 \cap C_2| \in \{0, 2, 3\}$.*

Proof: Case 1: $C_1 = a^*$ and $C_2 = b^*$. Since H is linear, at most one of its hyperedges can contain both a and b, therefore $|a^* \cap b^*| \leq 1$.

Case 2: C_1 and C_2 are both hole cliques. First note that every four hyperedges corresponding to vertices of some hole clique must contain all its seven points, and therefore determine the clique uniquely. Thus $|C_1 \cap C_2| \leq 3$, and the union of the corresponding hyperedhes contains at least 9 points. We assume that $x, y \in C_1 \cap C_2$ with hyperedges $X = \{a, b, c\}$ and $Y = \{c, d, e\}$ of H. Furthermore, we may assume that the further hyperedges corresponding to vertices in C_1 and C_2 are four or five of $\{a, e, f_1\}, \{a, d, g_1\}, \{b, e, g_1\}, \{b, d, f_1\}$, and $\{c, f_1, g_1\}$ respectively four or five of $\{a, e, f_2\}, \{a, d, g_2\}, \{b, e, g_2\}, \{b, d, f_2\}$, and $\{c, f_2, g_2\}$. By the linearity of H, $\{a, e, f_1\}$ and $\{a, e, f_2\}$ are either identical (i.e. $f_1 = f_2$) or not both present. By the same argument for the first four of the other possible hyperedges, at most 8 hyperedges occur. Thus $C_1 \cup C_2$ has at most 8 vertices, a contradiction.

Case 3: $C_1 = a^*$ and C_2 is a hole clique. Note that in the representation of C_2, every point b covered at all is covered by at least two (since $V(C_2) \geq 6$), but by at most three hyperedges. Thus $b^* \cap C_2$ contains two or three elements. QED

Theorem 7.10 *There is some $O(n^6)$-time algorithm that tests graphs of minimum degree at least 17 whether or not they are intersection graphs of linear 3-uniform hypergraphs.*

Proof: Let $G = \Omega(H)$ with H a linear 3-uniform hypergraph H, and let $\delta(G) \geq 17$. We may assume that G (and therefore H) is connected.

Although we will not, under our assumptions, be able to identify all star graphs among the cliques, we will at least be able to identify so many of them that every vertex is covered by one of them. Actually, we will clarify the status of every element of $\mathcal{M}_6$, where $\mathcal{M}_6$ denotes the set of cliques with at least 6 vertices. We compute that part $C_M(G)[\mathcal{M}_6]$ of the clique multigraph of G induced by $\mathcal{M}_6$.

Claim: There is at most one connected component of $C_M(G)[\mathcal{M}_6]$ without vertices corresponding to cliques with at least 8 vertices.

Let C_1 and C_2 be cliques in G with both at least 6 vertices, such that there are no paths in $C_M(G)[\mathcal{M}_6]$ from either C_1 nor C_2 to any vertex corresponding to a clique with at least 8 vertices. Let X be any hyperedge of H corresponding to a vertex of C_1, and let a be any point of X. In the same way, let b be a point in the representation of C_2, say in the hyperedge Y where $y \in V(C_2)$. Since H is connected, there is some sequence $a = a_0, a_1, a_2, \ldots, a_t = b$ such that each consecutive pair a_i, a_{i+1} is contained in some hyperedge X_{i+1}. We set $X_0 := X$ and $X_{t+1} := Y$.

We prove by induction that for every $0 \leq i \leq t+1$:

(a) every point of each X_i is covered by at most 7 and at least 6 hyperedges of H, and

(b) a_i^* lies in the same connected component of $C_M(G)[\mathcal{M}_6]$ than C_1.

For each i, the lower bound in (a) follows from the upper bound, since the number of neighbors of x_i in G equals the sum of the numbers of hyperedges covering the three points of X_i minus 3, and since $\delta(G) \geq 17$. This is the only point of the proof where we really need $\delta \geq 17$. For the upper bound and $i = 0$, if one point d of X_0 were covered by 8 or more hyperedges, then d^* would be a clique of G with at least 8 vertices, adjacent to C_1 in $C_M(G)$, a contradiction. (b) for $i = 0$ follows since C_1 and a_0^* are adjacent. For the induction step, validity of (a) for a given i follows by the validity of (b) for $i-1$: If a point d of X_i were covered by 8 ore more hyperedges, then d^* would be a clique of G containing 8 ore more vertices. But d^* is adjacent to a_{i-1}^*, and (b) for $i-1$ would imply that C_1 lies in the same component than a clique with at least 8 vertices in $C_M(G)[\mathcal{M}_6]$, contrary to our assumption. Validity of (b) for a given i follows by the validity of (a) for the same i and the validity of (b) for $i-1$.

Thus each one of the star graphs a_i^* contains at last 6 vertices. Therefore $C_1, a_0^*, a_1^*, \ldots, a_t^*, C_2$ is a path in $C_M(G)[\mathcal{M}_6]$, whence C_1 and C_2 are in the same connected component, and the claim is proven.

By Lemma 7.9, every component of $C_M(G)[\mathcal{M}_6]$ should allow a partition such that all edges between the classes have multiplicity 2 or 3, but inside each classes no multiple edges occur. If such a partition is not possible, then we are done—G is no such intersection graph in that case. Note also that the partition can be found by breadth first search in linear time. Let $\mathcal{M}_1' \cup \mathcal{M}_2'$ be that partition of those component (if there is any) $\mathcal{M}'$ whose vertices correspond to cliques with fewer than 8 vertices. Either all members of $\mathcal{M}_1'$ are star cliques and all members of $\mathcal{M}_2'$ not, or conversely. Both hypotheses have to be checked separately. But among the members of $\mathcal{M}_6 \setminus \mathcal{M}'$, the star graphs are uniquely determined, it is those partition class which contains the vertices corresponding to cliques with at least 8 vertices.

Since every vertex x has at least 17 neighbors which are covered by the three star graphs containing x, it must lie in some star graph with at least 6 vertices.

In the same way as described above we label the vertices by the number of star graphs with more than 5 vertices they lie in, delete all edges in those large star graphs, and test whether the remaining part is the intersection graph of some linear hypergraph with $S_x = \ell_x$, for every vertex x.

For the algorithm, we start by computing all K_4s $X_1, X_2, \ldots, X_t$ of G in time $O(n^4)$. Since intersection graphs of linear 3-uniform hypergraphs of are $(K_6 - e)$-free, no two cliques have four vertices in common. Thus every X_i must have a unique extension as a clique C_i. This C_i can be found by listing the set of all vertices S_i adjacent to all four vertices of X_i. We also have to check whether S_i is complete. This can be done in time $O(tn^2)$. Now note that $t \leq O(n^3)$, compare [P**]. Finding the bipartition of $C_M(G)[\mathcal{M}_6]$ and checking the consistency can be done in time $O(t^2) = O(n^6)$. So we try one of our two candidates for the star cliques among cliques with at least 6 vertices. We have to check how many of these cliques cover every vertex, this requires time $O(n^2 t) = O(n^5)$. Finally we delete the edges of these revealed star cliques, and test whether the remaining graph is a line graph in linear time QED

Problem 8 *What is the smallest integer μ for which recognizing intersection graphs of linear 3-uniform hypergraphs is still polynomial when restricted to graphs of minimum degree at least μ?*

8 Forbidden induced subgraphs

Recall that Scheinerman intersection classes are closed under induced subgraphs. Therefore there is always a characterization by forbidden induced subgraphs. There are only two difficulties with this approach: First, the complete list of these graphs may not be easy to get. Secondly, the list may be infinite. This could be fine, as for chordal graphs, for instance, but it may even be that the list lacks a finite description.

The nicest result in this direction is Beineke's famous characterization of line graphs by a list of nine small forbidden induced subgraphs.

For intersection graphs of 3-linear uniform hypergraphs of high enough minimum degree, there are also characterizations by forbidden induced subgraphs [NRSS82], [MT97]

But in most cases, all one could hope for is a list of *some* forbidden induced subgraphs. Although these lists do not lead to recognition, some of these forbidden induced subgraphs are rather important, like octahedrons, or graphs $K_n - e$, for instance, and should be routinely checked when dealing with classes of intersection graphs.

What about minors? By Robertson and Seymour's proof of the Wagner conjecture, every class of graphs closed under minors can be described by a *finite* list of forbidden minors. Unfortunately, few intersection classes are closed under minors—deleting an edge is not allowed, we cannot make two sets disjoint without affecting the intersections with the other sets.

On the other hand, many models behave quite well under contractions. Let $G = \Omega(A, (S_x/x \in V))$, and let $G_{/yz}$ be the graph obtained by contracting an edge yz. We get a representation of $G_{/yz}$ by letting all sets $S_v, v \neq y, z$ unchanged and taking $S_y \cup S_z$ for the new vertex. Therefore a Scheinerman class $\mathcal{G}(\mathcal{P})$ is closed under edge contraction if $S_y \cup S_z$ has again property $\mathcal{P}$ whenever S_x, S_y have property $\mathcal{P}$ and have nonempty intersection. Examples of such properties are being intervals of the real line, being subtree of a given tree, or being a connected subset of $\mathbf{R}^d$.

Such classes are closed under *induced* minors, where we only allow edge contraction and vertex deletion—edge deletion is forbidden. Unfortunately, the induced minor behave much worse than minors. There are classes closed under induced minors which cannot be described by a *finite* list of forbidden induced minors. Even the problem to test whether a given graph H is an induced minor could be NP-complete, for some fixed H[32]. Thus being closed under induced minors does not seem to help much.

9 Some NP-complete recognition problems

What if recognition is NP-complete? How could we prove it? Remember the splitting of the recognition problem into two subproblems in Subsection 4.1. If we could prove the hardness of the first step, i.e. the hardness of finding some list of graphs which could serve as star graphs, then the reconstruction problem is hard too, since every representation immediately yields star graphs. There might be cases, however, where we still don't know anything about the complexity of the of the *recognition* problem if the first step is hard. There might be other ways to recognize the graphs in question.

In general, hardness of the second subproblem doesn't imply hardness of the reconstruction of recognition problems, since the instances delivered by the first step might be tamer than expected.

Therefore, NP-completeness results should be easier to achieve in discrete models, where the second step, the layout step, is mostly trivial. This is indeed the case. I know of only one NP-completeness result for recognition of geometrically defined intersection graphs, namely boxes in $\mathbf{R}^d$, where $d \geq 3$. Roughly speaking, for most geometric models, the intersection graphs seem difficult to recognize but to prove the hardness of the problem seems also difficult.

Let us now treat some discrete models where in the first step of the approach it is NP-complete to decide whether suitable star graphs exist, but where step 2, expressed by some Krausz-type characterization, is trivial. Then recognition, as well as reconstruction, are NP-hard.

Let $\tau(G)$ denote the smallest integer for which a graph G is the intersection graph of some (not necessarily simple) k-uniform hypergraph.

Theorem 9.1 *[PRT81] Deciding whether $\tau(G) \leq k$ for instances G and k is NP-complete.*

Proof: We reduce reduction from the chromatic number problem. Let n be the number of vertices of G. We define $H = K_1 * (nK_1 \cup \overline{G})$, i.e. H is obtained by adding new vertices $y_1, y_2, \ldots, y_n, x$ to the complement[33] of G and joining x to all vertices of $V(G) \cup \{y_1, \ldots, y_n\}$. Using Proposition 7.1 one can show[34] $\chi(G) = \tau(G) - n$. QED

Problem 9 *Can τ be approximated?*

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[32] compare M.R. Fellows, J. Kratochvil, M. Middendorf, F. Pfeiffer, The complexity of induced minors and related problems, *Algorithmica* 13 (1995) 266-282.

[33] which has the same vertex set than G, but two distinct vertices are adjacent in $\overline{G}$ if and only if they are nonadjacent in G

[34] ...

Still the reduction doesn't say anything about the complexity of the problem $\tau(G) \leq k$ for $k \leq |V(G)|/2+2$, for instance. For fixed $k=3$ and simple hypergraphs we use a reduction from 3-SAT, variants of which seem to work in several of these situations.

Theorem 9.2 *[PRT81] Recognizing graphs in* $\mathcal{G}_0(3-\text{element sets})$ *is NP-complete.*

Proof: Let $c_1 \wedge \cdots \wedge c_m$ be an instance of 3-SAT, a boolean expression in variables $x, y, z \cdots$. We may assume that every clause contains exactly 3 distinct literals.

For every variable x we construct a graph H_x as in Figure 10. Some of its edges are labelled $x_1, \ldots, x_m, \overline{x_1}, \ldots, \overline{x_m}$.

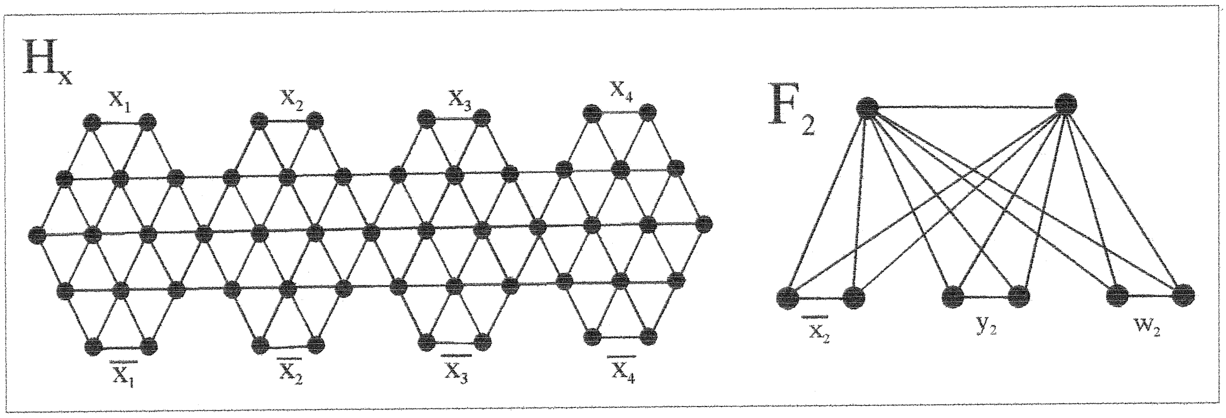

Figure 10

For every clause c_i we construct a graph F_i as in Figure 10 also, where three edges are labeled by the literals occuring in c_i. Now we take the disjoint union of all graphs and glue together edges with identical labels (in an arbitrary orientation), see Figure 11 for an example.

Now those coverings obeying the condition mentioned in Proposition 7.1 correspond 1-1 to those assignments of values to the variables $x, y, z, \ldots$ where $c_1 \wedge \cdots \wedge c_m$ is true. QED

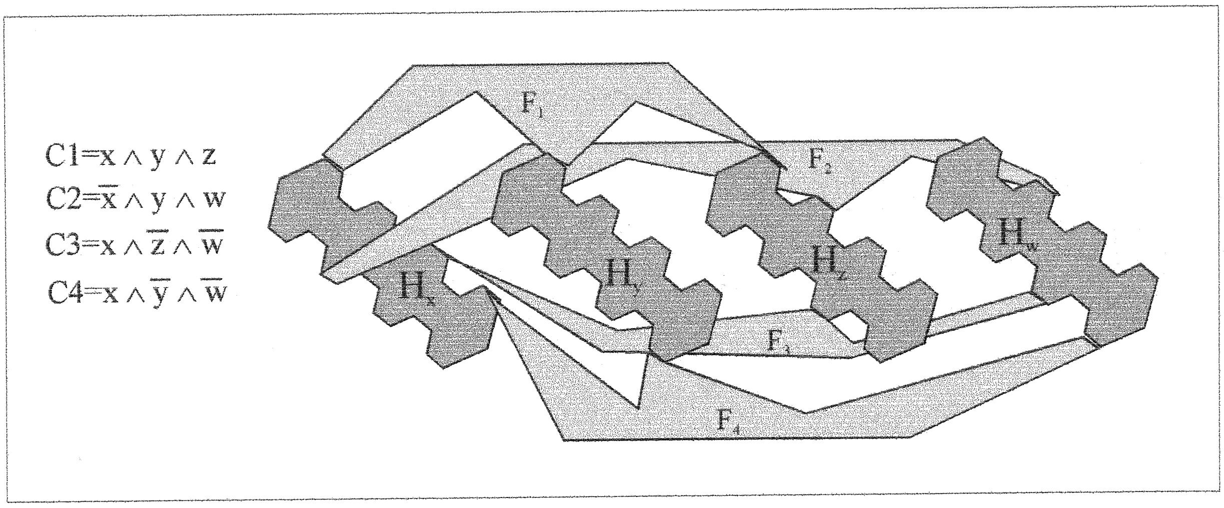

Figure 11: Putting the parts together

Although a similiar reduction shows that recognizing members of $\mathcal{G}(4-\text{element sets})$ is also NP-complete, the complexity of recognizing members of $\mathcal{G}(3-\text{element sets})$ (without requiring simplicity of the hypergraph, that is) is still unsettled.

For intersection graphs of 3-uniform *linear* hypergraphs there is a similiar reduction. Maybe it suffices if I show you the ingredients. The graphs used are given in Figure 12.

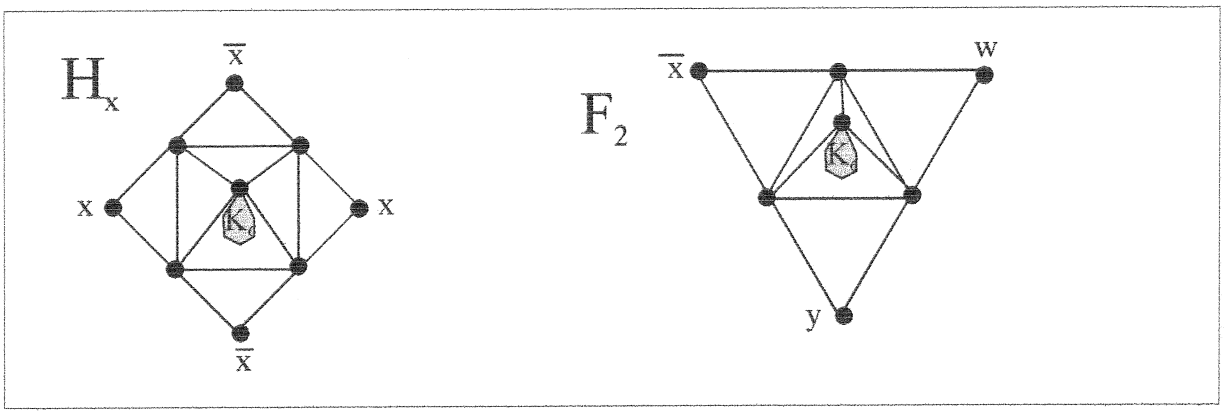

Figure 12

The possible coverings of these parts are given in Figure 13.

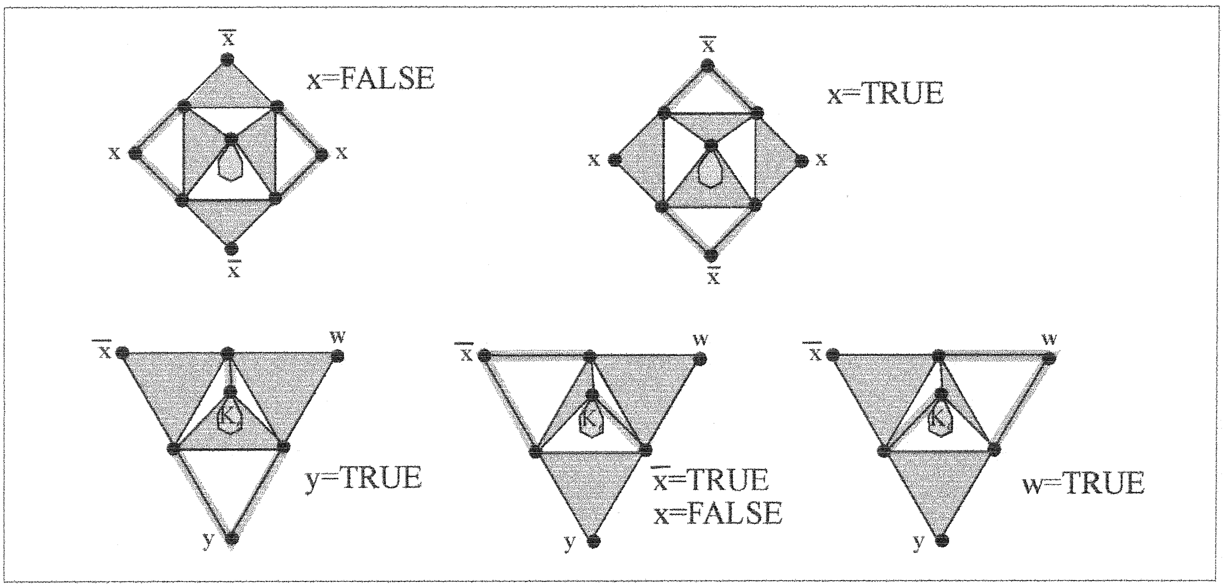

Figure 13

Finally these blocks are glued together like shown in Figure 14.

Theorem 9.3 *[HK97] Recognizing intersection graphs of linear 3-uniform hypergraphs is NP-complete.*

The construction also shows that even for graphs of minimum degree at least 4 (and even 5, provided 3-SAT restricted to instances, where every x and every $\overline{x}$ occurs in at least 4 clauses, is still NP-complete) the problem is hard. Thus the μ in Problem 7 could not be smaller than 5 (respectively 6).

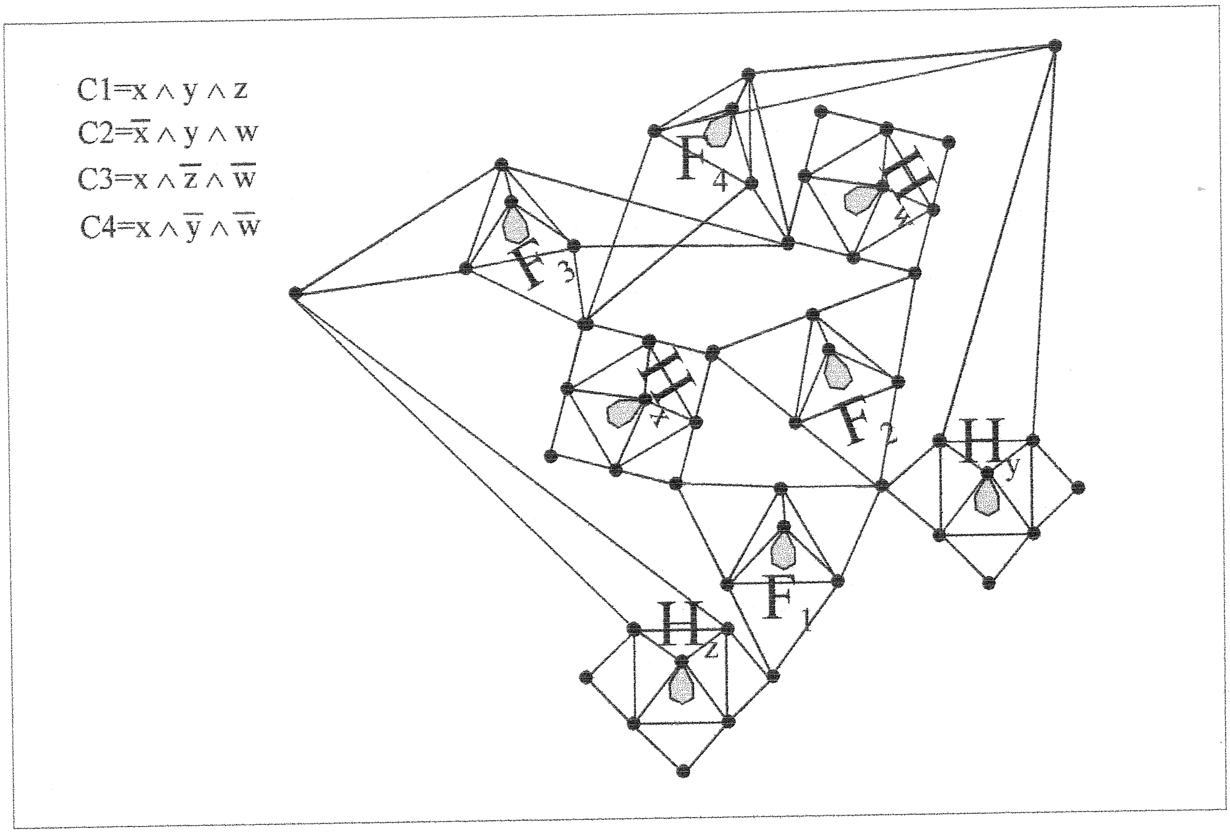

Figure 14: Glueing the parts this time together

10 Betti numbers of intersection graphs

Under certain conditions, the topology of the union of the hyperedges is to some extend reflected in the structure of its intersection graph. This is the reason why the octahedron is no intersection graph of rectangles in the plane, but of unit disks, for instance.

When speaking of the topology of a graph, the 1-dimensional simplicial complex is not relevant for our purposes, instead let us define the simplicial complex $\widehat{G}$ of a graph to be the complex whose simplices are the vertex sets of all complete subgraphs of G. We compare the topology of this complex with the topology of the union of all sets in the geometric case. In the discrete case, we define the complex $\tilde{H}$ of a hypergraph $H = (A, (S_x/x \in V))$ to have all subsets of the hyperedges as simplices.

The connection between the topologies is not too astonishing, since there is some connection between the union of these geometric sets, respectively $\tilde{H}$, to the *nerve* of the hypergraph, which is just $\tilde{H}^*$. This follows by classical topological theorems. On the other hand, $\widehat{G}$ is just a subcomplex of this nerve, and sometimes they even coincide.

Lemma 10.1 *$\tilde{H}^* = \widehat{G}$ if and only if H obeys the Helly property.*

We distinguish these topological spaces by means of their (modulo 2) Betti numbers, i.e. ranks of the modulo 2 homology groups. You may think of β_0 counting the connected components of the space, β_1 counting the number of independent 'tunnels' through the space, β_2 counting the number of independent 'Swiss cheese holes', and so on.

Let us illustrate these concepts: The complexes of the octahedron graphs are homotopic to the spheres, we have $\beta_i(\widehat{nK_2}) = 1$ just for $i = 0, n-1$, and $= 0$ for all other i.

10.1 geometric models

A system of subsets of $\mathbf{R}^d$ has the *Leray-property* if every nonempty intersection is homologically trivial (i.e. has $\beta_0 = 1$, but $0 = \beta_1 = \beta_2 = \cdots$). For many properties $\mathcal{P}$, the Leray-property is automatically fulfilled f or every system of $\mathcal{P}$-sets. Examples are straight lines, or unit-disks, or boxes.

The key for homology for such families lies in the following classical result of LERAY:

Theorem 10.2 *[L45] Nerve and union are homological equivalent for every family of subsets of $\mathbf{R}^d$ obeying the Leray-property.*

Corollary 10.3 *Let G be the intersection graph of a family $(S_x/x \in V)$ of subsets of $\mathbf{R}^d$ obeying both the Helly- and the Leray-property. Then $\widehat{G}$ and $\bigcup_{x\in V} S_x$ are homologically equivalent. In particular $0 = \beta_d(\widehat{G}) = \beta_{d+1}(\widehat{G}) = \cdots$ and $\overline{(d+1)K_2}$ is not an induced subgraph of G.*

Proof: The first statement follows by Theorem 10.2 and Lemma 10.1. The second follows immediately since higher homology groups of subsets of $\mathbf{R}^d$ vanish. QED

Therefore, for every property $\mathcal{P}$ where the Helly- and Leray-properties are both true for every family of $\mathcal{P}$-sets, computing maximum cliques in $\mathcal{G}(\mathcal{P})$ is polynomial. An example is the class of intersection graphs of boxes.

Corollary 10.4 $0 = \beta_1(\widehat{G}) = \beta_2(\widehat{G}) = \cdots$ *for every chordal graph G.*

Proof: Subtrees of some tree obey both the Helly- and the Leray-property. By Corollary 10.6, $\widehat{G}$ is homologically equivalent to some substructure of the tree. QED

Problem 10 *What happens if only relaxed versions of the Leray-property are valid: Either we have $0 = \beta_k = \beta_{k=1} = \cdots$ for every nonempty intersection of S_xs, for some fixed integer k, or we request $k \geq \beta_0, \beta_1, \beta_2, \ldots$ for every nonempty intersection of S_xs, again for some fixed k?*

10.2 discrete models

For discrete models, we only need the Helly-property. The Leray-property is, in a sense, automatically fulfilled.

Theorem 10.5 *[D52] For every hypergraph H, $\tilde{H}$ and $\tilde{H}^*$ are homological equivalent.*

Corollary 10.6 *Let G be the intersection graph of a Helly-hypergraph $(A, (S_x/x \in V))$. Then $\widehat{G}$ and $\tilde{H}^*$ are homologically equivalent. In particular $0 = \beta_r(\widehat{G}) = \beta_{r+1}(\widehat{G}) = \cdots$, where $r = \max_{x\in V} |S_x|$, and $\overline{(d+1)K_2}$ is not an induced subgraph of G.*

Without the Helly property, not too much is known. Let us concentrate on intersection graphs of simple k-uniform hypergraphs again. For $k = 2$ we have an almost complete picture:

Theorem 10.7 *[P91] For every graph G, $\beta_1(\widehat{G}) = \beta_1(\widehat{L(G)})$, $\beta_2(\widehat{G}) \leq \beta_2(\widehat{L(G)})$, and $\beta_i(\widehat{L(G)}) = 0$ for all $i \geq 3$.*

Unfortunately nothing is known for higher k. Since large enough octahedron graphs are forbidden by Theorem 7.3, we might ask:

Problem 11 *For which $k \geq 2$ is there some $f(k)$ such that every intersection graph G of a simple k-uniform hypergraph obeys $\beta_i(\widehat{G}) = 0$ for every $i \geq f(k)$?*

Certainly $f(2) = 3$, and also $f(k) \geq \binom{2k-1}{k}$, if existing at all.

11 ℓ-intersection

The *ℓ-intersection graph* $\Omega_\ell(H)$ of a hypergraph $H = (A, S_x, x \in V)$ has V as vertex set, and two distinct vertices x, y are joined by an edge whenever $|S_x \cap S_y| \geq \ell$.

This concept has been introduced in [BHS77] and [HS76]. ℓ-intersection makes only sense for discrete intersection models—the appropriate generalization for geometric models would be tolerance intersection.

The difficulty of the recognition problem seems higher than for ordinary intersection, since, as we shall see, in the duality approach, we have a third step to tackle. Instead of the points of the hypergraph, now ℓ-element point sets serve as detectors. For every such set M of $\binom{A}{\ell}$, we define M^* as the set of those $v \in V$ for which $M \subseteq S_v$. Obviously each such M^* induces a complete graph in the ℓ-intersection graph G. Moreover, every edge of G is detected by some ℓ-set of A. If we denote these complete graphs M^* as the *star graphs* of G, then

the set of star graphs forms an edge cover by complete graphs of G.

	1	2	3	4	5	6	7	8	9	10	11
M_1	*	*	*								
M_2	*			*					*		
M_3			*					*			*
M_4				*	*	*					
M_5						*	*	*			
M_6									*	*	*

Figure 15: A hypergraph which is not a 2-tuple hypergraph.

For integers $\ell \geq 1$ and a hypergraph $H = (A, (S_x/x \in V))$, let its *ℓ-set hypergraph* $H[t]$ denote the hypergraph with those elements of $\binom{A}{\ell}$ as vertices that occur in some hyperedge of H, and $(S_x[\ell]/x \in V)$ as hyperedge family, where $S_x[\ell]$ contains all sets $B \in \binom{A}{\ell}$ for which $B \subseteq S_x$. Now ℓ-intersection graphs are just special intersection graphs:

Proposition 11.1 $\Omega_\ell(H) = \Omega(H[\ell])$.

If H is t-linear (i.e. every two hyperedges have at most t points in common) and k-uniform, then $H[\ell]$ is $\binom{k}{\ell}$-uniform, and $\binom{t}{\ell}$-linear. Therefore we get a generalization of a result in [LP*] and [JKL96] for $\ell = t = k - 1$

Proposition 11.2 *ℓ-intersection graphs of k-uniform t-linear hypergraphs are intersection graphs of certain $\binom{k}{\ell}$-uniform, $\binom{t}{\ell}$-linear hypergraphs.*

For $\ell \geq 2$, not every $\binom{k}{\ell}$-uniform, $\binom{t}{\ell}$-linear hypergraph is the ℓ-set hypergraph of some hypergraph, as can be seen in the example of Figure 15.

11.1 k-facet graphs

You remember that no analogue of Lemma 4.3 was possible for intersection graphs of k-uniform hypergraphs and $k \geq 3$? Actually no such result is possible for ℓ-intersection graphs of simple k-uniform hypergraphs and $\ell \leq k - 2$. Only for $\ell = k - 1$ it works:

Lemma 11.3 *[LP*] Let $k \geq 2$, and let the k-sets $X_1, X_2, \ldots, X_t$ have pairwise $k - 1$ elements in common. Then $|\bigcap_{i=1}^t X_i| = k - 1$ or $|\bigcup_{i=1}^t X_i| = k + 1$.*

Therefore, every clique of size at least $k+2$ in the $k-1$-intersection graph of some simple (!) k-uniform hypergraph is a star graph. This should justify to give this concept a try. $(k-1)$-intersection graphs of simple k-uniform hypergraphs are abbreviated as *k-facet graphs*. Recall that every k-facet graph is the intersection graph of some linear k-uniform hypergraph, see Proposition 11.2.

Star graphs can be revealed for $k=2$, compare Subsection 6.2, but not for $k \geq 3$. See Figure 16 for an example.

However, the ideas used in there *almost* carry over: Let S_x and S_y be hyperedges of the k-uniform hypergraph H with $|S_x \cap S_y| = k-1$. In the k-facet graph, the edge xy can be extended in only one way to a star graph clique, namely by adding all vertices z whose set S_z includes $S_x \cap S_y$. Also, since $|S_x \cup S_y| = k+1$, by Lemma 11.3 there is at most one way to extend xy to some non star graph clique, namely by adding all z whose S_z is contained in $S_x \cup S_y$. Therefore, two star graphs in a k-facet graph have at most one common vertex, and two non star graph cliques have also at most one common vertex. Unfortunately, a star graph and a non star graph clique may have 0,1, or 2 common vertices, so there remains some ambiguity[35].

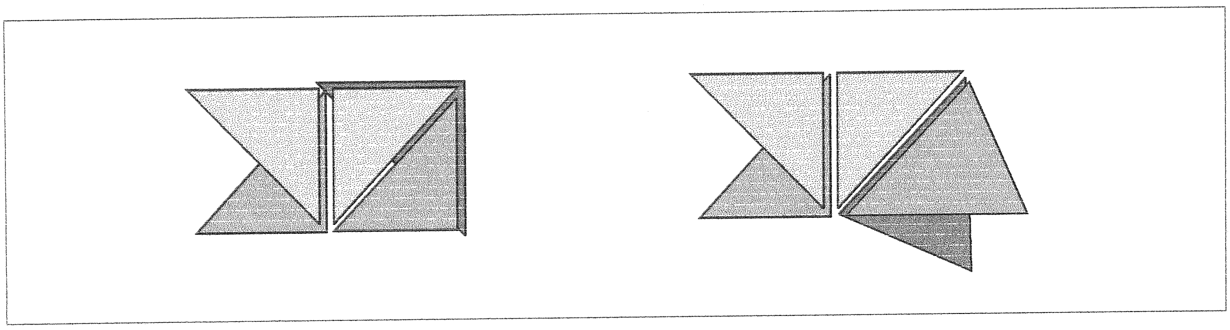

Figure 16:

There is at least a related problem where the star graph identification step works. The *k-line graph* of a graph is the $(k-1)$-intersection graph of the set of all K_ks. In other words, k-line graphs are just the k-facet graphs of k-uniform k-conformal[36] hypergraphs. Star cliques and non star graph cliques have 0 or 2 common vertices, thus again we have only two possibilities for the set of star graphs (since $C_M(G)$ is again connected).

The general problem is hard:

Theorem 11.4 *[JKL?] For every $k \geq 3$ it is NP-complete to decide whether a triangle-free graphs is a k-facet graph.*

Whether this difficulty is due to the difficulties of revealing star graphs or to the difficulty of recognizing ℓ-set hypergraphs, I do not know.

Problem 12 *What is the complexity of the problem to recognize ℓ-set hypergraphs of $(\ell+1)$-uniform hypergraphs, for $\ell \geq 2$?*

Problem 13 *What is the complexity of recognizing k-line graphs?*

Problem 14 Flags *in a partially ordered set (poset) are inclusion-maximal chains. A poset is* k-pure *if all flags have the same cardinality k. The* flag hypergraph *of a k-pure poset $P = (A, <)$ is the hypergraph whose hyperedges are all flags of P. The $(k$-$)$* flag graph *of P is the facet graph of the hypergraph of all flags. What is the complexity of recognizing k-flag graphs*[37] *for $k \geq 3$?*

[35] But read on in the next section.

[36] if every pair of vertices in a k-element set S is covered by some hyperedge, then this set is a hyperedge

[37] 2-flag graphs are just the line graphs of bipartite graphs, characterized in S.T. Hedetniemi, Graphs of $(0,1)$-matrices, LNM 186 (1971) 157-171.

12 Random models

It is certainly convincing to consider intersection graphs of random models, but one has to be careful in defining what a random set (of a certain shape) should be. For instance, how would you define a random interval, or a random unit disk? The given distribution should also make sense, for instance in applications.

For instance, a model for random interval graphs has been proposed in [JSW90]. Choose n pairs of points, which then are the endpoints of the intervals, from a given continuous distribution on the line. It is easy to see that this distribution actually doesn't matter. The authors derived interesting properties of the resulting random interval graph, namely that the expected clique size is $n/2$, the expected independence number $2\sqrt{n/\pi}$, and the resulting graph has some vertex adjacent to all others with probability $2/3$. Thus in most cases we get diameter 2.

Problem 15 *Are there other reasonable models for random interval graphs? What about the model for random unit interval graphs, where we choose n unit intervals out of $[0,m]$, with uniform distribution?*

Like general random graphs, intersection graphs of random models seem to have very sharp features, as can be seen in the example above. Another example is the following: Among the set of all 2^N subsets of $\{1,2,\ldots,N\}$, choose $n(N)$ independent random subsets. Then the intersection graphs $G(n.N)$ have the following properties:

Theorem 12.1 *[M91] For the model described above,*

- *if $n \prec (4/3)^{N/2}$, then $G(n,N)$ is complete almost surely*[38] *,*
- *if $(4/3)^{N/2} \prec n \prec 2^N$, then $G(n,N)$ is connected with diameter 2, almost surely, and*
- *if $2^N \prec n$, then $G(n,N)$ contains isolated vertices almost surely.*

Thus this model wouldn't be useful for getting connected graphs of large diameter, for instance. It should be worthwhile to try intersection graphs of which models have which properties[39]

12.1 A recognition hierarchy

Remember the cases where we had only partial results for recognition. Some of them had the type that large enough minimum degree (i.e. minimum $|a^*|$) required for the model made things going, see Proposition 7.8. In others, we could handle all graphs of large enough minimum degree, see Theorem 7.10.

A result of the latter shape is surely preferable, since if we require large enough minimum degree in the model, then we also get large minimum degree in the graph. Now every result requiring large minimum degree of the hypergraph works in almost all cases in most models, so we get the following hierarchy, going from weak to strong results:

1. Recognition in almost all cases.
2. Recognition for models of high enough minimum degree.
3. Recognition for graphs of high enough minimum degree.
4. Recognition in all cases.

[38] 'almost surely' refers to N going to ∞

[39] for instance, k-facet graphs of random k-uniform hypergraphs $\mathcal{H}_k(n,p)$ have diameter k almost surely for $p = n^{-1/2+\varepsilon}$, compare E. Prisner, J. Rudolph, Random interconnection networks, in preparation.

12.2 k-facet graphs revisited

Let $k \geq 2$ be a fixed integer. Random simple k-uniform hypergraphs are easy and natural to define, just choose some function $p(n)$ with $0 \leq p(n) \leq 1$, and let $\mathcal{H}_k(n,p)$ be the random hypergraph with vertex set $\{1,2,\ldots,n\}$ where each k-element subsets has independently the same chance $p(n)$ to occur as hyperedge.

We shall show that for suitable $p(n) = n^{-1/2+\varepsilon}$, $\varepsilon > 0$, k-facet graphs of $\mathcal{H}_k(n,p)$ are almost always recognizable, for $n \to \infty$. Each of both remaining problems—identifying the star graphs and recognizing $(k-1)$-set hypergraphs of k-uniform hypergraphs—becomes tractable almost always.

For the first we show that for almost every such k-facet graph, all its star cliques have more than $k+1$ vertices. So take any fixed $(k-1)$-element set S. The probability that $k+1$ or less hyperedges contain S is

$$(1-p)^{n-k+1} + (n-k+1)(1-p)^{n-k}p + \binom{n-k+1}{2}(1-p)^{n-k-1}p^2 + \cdots$$
$$+\binom{n-k+1}{k+1}(1-p)^{n-2k}p^{k+1} = (1-p)^{n-2k}O(n^{(k+1)(\varepsilon+1/2)})$$

For the $\binom{n}{k-1}$ possible sets S, these probability are not independent, but the probability that there is a bad S, where the corresponding star graph S^* has fewer than $k+2$ vertices, is at most $\binom{n}{k-1}(1-p)^{n-2k}O(n^{(k+1)(\varepsilon+1/2)})$, a term converging to 0 for $n \to \infty$. Actually we really needed only a smaller hyperedge probability p here.

For the other part, testing whether a given simple, linear k-uniform hypergraph H' is the $(k-1)$-set hypergraph of some simple k-uniform hypergraph H, we try the following approach: Assume $H' = H[k-1]$. Under this assumption, in the first step we try to find out which pairs of vertices of H' have—viewed as $k-1$-element subsets of $A(H)$—$k-2$ common elements. These pairs are sampled as edges of the graph $F = (A(H'), E)$. If this has been done, we only need to check in the second step whether the resulting graph is a $(k-2)$-intersection graph of some $(k-1)$-uniform hypergraph. This second step sounds difficult enough, but for $k=3$ we can do it; it is just recognizing line graphs.

If two distinct $(k-1)$-element subsets are both contained in the same k-element superset, then we have exactly $k-2$ common elements. Thus

> (1) all pairs of vertices lying in the same hyperedge of H' are included in E.

However, we may have to include more.

Next assume that the vertices x and y of H' are in no common hyperedge, but that another vertex z joins a common hyperedge with x and another common hyperedge with y. If X, Y, and Z are the corresponding $(k-1)$-element subsets of $A(H)$, then $|X \cap Z| = k-2$ and $|Y \cap Z| = k-2$, thus $|X \cap Y| \geq k-3$. If $|X \cap Y| = k-3$, at most four such vertices z are possible. Thus, if there are at least five such vertices z, then we know $|X \cap Y| = k-2$ and add xy to our edge set E.

> (2) If there are distinct vertices $x, y, z_1, z_2, z_3, z_4, z_5$ in H' where for every $1 \leq i \leq 5$ some hyperedge of H' contains both x and z_i, and another hyperedge contains both y and z_i, then we add xy to E.

There may be, however, still more edges in E.

Not so if H is random, $H = \mathcal{H}_k(n,p)$, with large enough n and high enough edge probability p.

Proposition 12.2 *Let $H' = \mathcal{H}_k(n,p)[k-1]$, where $p = p(n) = n^{-1/2+\varepsilon}$, for $\varepsilon > 0$. Then, almost surely, the graph F constructed by the two rules (1) and (2) above, is connected, and F is the $(k-2)$-intersection graph of the $(k-1)$-uniform hypergraph H'' whose hyperedges are all $(k-1)$-element subsets of the hyperedges of $\mathcal{H}_k(n,p)$.*

Proof: What is the probability that two fixed $(k-1)$-element subsets X and Y have $k-2$ common elements but xy is not in included by rules (1) and (2)? Of the $n-k$ independent events whether both $X \cup \{z\}$ and $Y \cup \{z\}$ are hyperedges of $\mathcal{H}_k(n,p)$, for $z \notin X \cup Y$, having each probability p^2, at most four should occur. The probability for that is

$$(1-p^2)^{n-k} + (n-k)(1-p^2)^{n-k-1}p^2 + \cdots + \binom{n-k}{4}(1-p^2)^{n-k-4}p^8 = (1-p^2)^{n-k-4}O(n^{8\varepsilon}).$$

There are at most $k\binom{n}{k}$ such pairs X, Y of $(k-1)$-element sets with $k-2$ common vertices. These events of $xy \notin E$ are not independent, but the probability that F does not contain all edges of the $(k-1)$-facet graph is at most

$$\begin{aligned}\mathcal{P}(F \neq \Omega_{k-2}H'') &\leq k\binom{n}{k}(1-p^2)^{n-k-4}O(n^{8\varepsilon}) \leq \\ &\leq O(n^{k+8\varepsilon})(1-\frac{n^{2\varepsilon}}{n})^{n-k-4} \to O(n^{k+8\varepsilon})e^{-n^{2\varepsilon}}.\end{aligned}$$

This expression goes (very slowly) to 0 for $n \to \infty$.

Connectedness of F is also straightforward to show. QED

For recognizing 3-facet graphs of large random 3-uniform hypergraphs we simply decide that all cliques with more than 4 vertices are star graphs. We test whether this hypergraph is the 2-set hypergraph of some 3-uniform hypergraph, and are done. The only problem is that the hypergraph of the star graphs is not really random, but it can be shown that it behaves random. For testing k-facet graphs with $k \geq 4$, we do the same and arrive at the problem of recognizing $(k-1)$-facet graphs of $\mathcal{H}_{k-1}(n,1)$, which works quite well.

13 Intersection number

It seems to be a natural goal to represent graphs as intersection graphs of (simple) hypergraphs with minimum vertex set. The *intersection number* $\Theta(G)$ of a graph is this minimum number, i.e. the smallest n for which G is the intersection graph of some hypergraph $(A, (S_x/x \in V))$ with $|\bigcup_{x \in V} S_x| = n$.

Theorem 13.1 *[EGP66] The edge set of every n-vertex graph without isolated vertices can be partitioned into $\lfloor n^2/4 \rfloor$ edges or triangles.*

Proof: We use induction on n. Let G have n vertices, none of them isolated. Since $\lfloor n^2/4 \rfloor = \lfloor (n-1)^2/4 \rfloor + \lfloor n/2 \rfloor$, we only have to show how such a partition for some $(n-1)$-vertex subgraph G' of G (which has to exist, by induction hypothesis) can be extended to some partition for the whole graph.

If G has some vertex x of degree $\leq \lfloor n/2 \rfloor$, this is obvious—we choose $G' = G - x$ and add all edges incident with x.

So assume in the following $\delta(G) = \lfloor n/2 \rfloor + r$, where $r > 0$, and choose a vertex x where $d_G(x) = \delta(G)$.

We show that $G[N(x)]$ must have r independent edges $e_1, \ldots, e_r$. For, otherwise every neighbor y of x not occuring in any of these edges has at most $2r-2$ neighbors in $G[N(x)]$, and obviously at most $n - d_G(x)$ outside. Thus

$$d_G(y) \leq 2r - 2 + n - (\lfloor n/2 \rfloor + r) < \lfloor n/2 \rfloor + r,$$

a contradiction.

Now obtain G' from G by first deleting x and then $e_1, \dots, e_r$. To the existing partition of the edge set of G', we add all r triangles formed by x and the e_i, and we add all the remaining $\lfloor n/2 \rfloor - r$ edges incident with x. QED

The result is sharp, as can be seen by the complete bipartite graphs $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Using dualization, there follows that every graph is the intersection graph of some *linear* hypergraph with at most $\lfloor n^2/4 \rfloor$ vertices. If there are hyperedges of cardinality 1, then this hypergraph is not necessarily simple, but it is also possible to express G as intersection graph of some *simple* hypergraph with at most $\lfloor n^2/4 \rfloor$ vertices.

Theorem 13.2 *[KSW78] Finding a minimum covering of the edges of a graph by complete subgraphs is NP-complete.*

Proof: Given a graph G, the minimum cardinality $\overline{\chi}(G)$ of a set of complete subgraphs covering all *vertices* of G is the chromatic number of the complement $\overline{G}$ of G, and is therefore NP-complete.

Let G' be constructed from G by adding $m+1$ new vertices $a_1, a_2, \dots, a_{m+1}$, all adjacent to the vertices of G, where m is the number of edges of G. Let $S_1, S_2, \dots S_k$ be a minimum covering of the vertices of G by complete subgraphs. Then all $k(m+1)$ complete graphs $S_i \cup \{a_j\}$, together with all edges of G cover all edges of G', therefore

$$\Theta(G') \leq (m+1)\overline{\chi}(G) + m.$$

Now assume we had some polynomial-time algorithm for finding minimum covering of the edges by complete graphs for every graph. Then we find such a cover for G'. For every $1 \leq i \leq m+1$, let M_i the set of those members of our cover containing a_i. Since the members of each M_i cover all edges from a_i to vertices of G, they cover all vertices of G, whence $|M_i| \geq \overline{\chi}(G)$ for every $1 \leq i \leq m+1$. We claim $\min_{1 \leq i \leq m+1} |M_i| = \overline{\chi}(G)$, a contradiction to the NP-completeness for computing $\overline{\chi}(G)$. For, assume $|M_i| > \overline{\chi}(G)$ for every $1 \leq i \leq m+1$. Then our optimal cover contains at least $(m+1)(\overline{\chi}(G)+1)$ elements, a contradiction to the formula above. QED

Using essentially the same idea, but adding $Ce+1$ new vertices instead of $e+1$, it is possible to show that, for every fixed constant $C \geq 1$, every algorithm that always finds a covering of the edges of every graph G of cardinality at most $C\Theta(G)$, could be used to find a coloring of every graph G using at most $C\chi(G)+d$ colors. But such a polynomial algorithm would imply $P = NP$ by [LY93] and could therefore not be expected to exist.

Concerning random graphs $\mathcal{G}(n, 1/2)$

$$(1-\varepsilon)\frac{n^2}{(2\ln n)^2} \leq \Theta(G) \leq O(\frac{n^2 \ln\ln n}{(\ln n)^2})$$

for almost every graph [BESW93].

In the same way, the *p-intersection number* $\Theta_p(G)$ of a graph G is the smallest vertex set of a hypergraph whose p-intersection graph equals G. Concerning the graphs $K_{n,n}$ it seems like $\Theta_p \approx \Theta/p$, since $\Theta_p(K_{n/2,n/2}) = \frac{n^2}{4p}(1+o(n))$ [EGR96], [F97].

14 Distance

In Section 10 we saw that topological properties of the representation may be reflected in the intersection graph. Even metric properties, like distance, somtimes show through.

Why is every power of an interval graphs an interval graphs, but not every power of a unit interval graph a unit interval graph? Why are odd powers of chordal graphs chordal, but the even powers not necessarily?

Let G be the intersection graph of the hypergraphs $H = (A, (S_x / x \in V))$. Then distances in G can be expressed by set intersection of the sets S_x^t, where $S_x^1 := S_x$ and S_x^t is recursively defined[40] as the union of S_x^{t-1} and those S_z intersecting S_x^{t-1}. We get $d_G(x,y) \leq i+j-1$ if and

[40] note that we also have $S_x^t := \bigcup_{y/d_G(x,y) \leq t-1} S_y$.

only if $S_x^i \cap S_y^j \neq \emptyset$. Therefore $(A, (S_x^t/x \in V))$ is a intersection representation of the $(2t-1)$th power of G, whose vertices are adjacent if and only if $d_G(x,y) \leq 2t$. In some situations, these sets S_x^t have the same shape than the original sets S_x. For instance, if the only property $\mathcal{P}$ the sets S_x have to be obey is, that they should be connected subsets of some topological space, then the sets S_x^t are also connected subsets of it. Then every odd power of a graph in $\mathcal{G}(\mathcal{P})$ lies again in $\mathcal{G}(\mathcal{P})$. A prominent example is the classe of chordal graphs

Theorem 14.1 *[BP83], [D84] Odd powers of chordal graphs are chordal.*

But we get an odd power result also for circular-arc graphs or interval graphs in this way.

What about even powers? Here we need geometric models with ordering, compare Section 5. Assume G is the intersection graph of $\mathcal{P}$-sets S_x, where for every two disjoint sets, one is to the left of the other. Assume furthermore that $S_y \setminus S_x$ divides naturally into two parts, one to the left and the other to the right of S_x. Then we may define R_x as the union of S_x and all right parts of $S_y \setminus S_x$ for neighbors y of x. It turns out that these sets R_x are an intersection representation of G^2. The only problem may be that these sets are not necessarily $\mathcal{P}$-sets. In some cases, however, we may replace them by $\mathcal{P}$-sets having the same intersection pattern than the sets R_x.

If the S_x^t-construction above applies also, we get representations of all powers of G by $\mathcal{P}$-sets in that case.

This approach works for intervals on the real line without further ado. It works for trapezoids between two parallel lines by taking the convex hulls of the sets R_x constructed (which are no trapezoids). It even works for structures with some cyclic ordering, as for connected subsets (circular arcs) of the unit circle.

Theorem 14.2 *(a) [R87] All powers of interval graphs are interval graphs.*

(b) [F95] All powers of trapezoid graphs[41] *are trapezoid graphs.*

(c) [D92] All powers of cocomparability graphs are cocomparability graphs.

(d) [R92] All powers of circular-arc graphs are circular-arc graphs.

Problem 16 *A. Raychaudhouri posed the problem whether or not it is true that whenever G^k is a circular-arc graph, then G^{k+1} must too be a circular-arc graph*[42].

There are several other situations where the distances transfer, let me just mention one more example:

Theorem 14.3 *[H84], [H86] For every graph G, $diam(G) - 1 \leq diam(L(G) \leq diam(G) + 1$, as well as $diam(G) - 1 \leq diam(C(G) \leq diam(G) + 1$.*

15 Intersection multigraphs

In the *intersection multigraph* $\Omega_M(H)$ of a hypergraph $H = (A, (S_x/x \in V))$, vertices $x \neq y$ are connected by $|S_x \cap S_y|$ parallel edges. As ℓ-intersection, it mainly makes sense in discrete models. Intersection multigraphs contain all the information of all ℓ-intersection graphs.

As mentioned in Section 6.2, it is presently unknown how to recognize clique multigraphs of chordal graphs, whereas clique graphs of chordal graphs are efficiently recognizable. In this section I will present an example for the contrary effect, where intersection multigraphs seem easier to recognize, and one example where both are doable.

[41] intersection graphs of trapezoids between two fixed parallel lines

[42] This is true for *proper* circular-arc graphs, compare E. Prisner, A note on powers and proper circular-arc graphs, JCMCC 1997

Let us start with so-called chordal multigraphs, intersection multigraphs of subtrees of some tree. This essentially geometrical model could also be interpreted as discrete, and in many applications, trees are rather discrete than continuous, and then the notion of intersection multigraphs fits.

For an edge xy in some multigraph, let $\mu(xy)$ denote the number of parallel edges from x to y, and let $\kappa(xy)$ denote the number of cliques of the underlying graph containing edge xy.

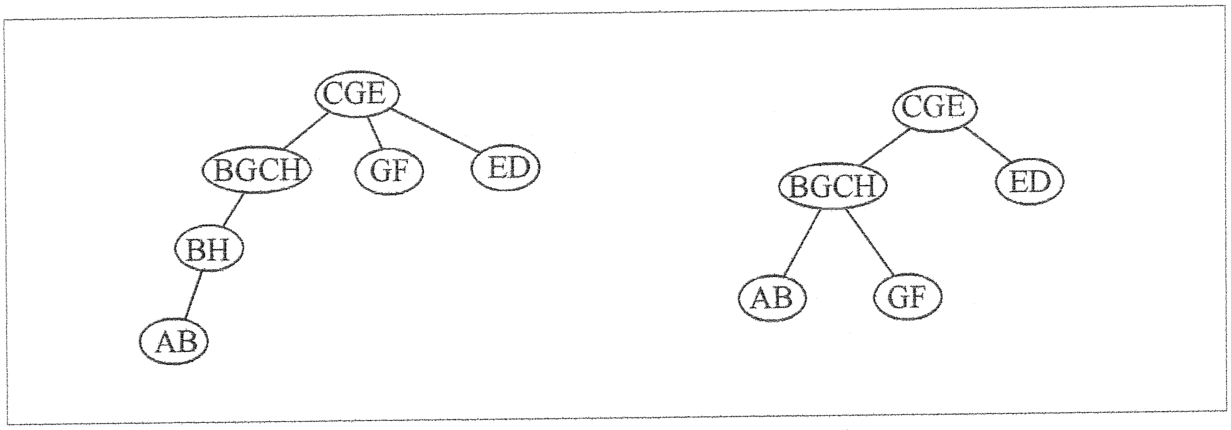

Figure 17: Two possible phylogenetic trees for Example 7, see Figure 3.
The second one is only possible if we do not know multiplicities of the intersections.

Theorem 15.1 *[Mc91] A multigraph is chordal if and only if its underlying graph is chordal and it fulfills the $\mu \geq \kappa$ condition.*

Proof: Let M be the intersection multigraph of the family $(T_x/x \in V)$ of subtrees of some given tree T. Other than for ordinary intersection graphs, we can *not* require a^* to be a clique (of the underlying graph G of M) for every vertex a of T. But some of them are, and every clique of G must be such a a^*, by the Helly property. Therefore, if two vertices lie in $\kappa(xy)$ common cliques, then the corresponding subtrees T_x and T_y both contain at least $\kappa(xy)$ common vertices, thus $\mu(xy) \geq \kappa(xy)$.

For the other direction, we extend M to some multigraph M' whose underlying graph is again chordal, but where $\kappa = \mu$. This is simply done by repeatedly adding new vertices, and making them simply adjacent to the two endvertices of an edge xy where $\mu(xy) < \kappa(xy)$.

Next we choose any representation of the underlying graph G' of M' having the property that all vertices of the tree correspond to cliques of G'. As mentioned, compare also Theorem 6.3, such a representation is always possible. It is easy to see that this is also a representation of M'.

Finally, by deleting representatives of the added vertices, one surely gets a representation of M. QED

By a construction like in the proof, one gets the representation in Figure 17a of the multigraph in Figure 3; Figure 17b is just a ordinary intersection representation of the underlying graph. Compared with this, in the multigraph model we detect that there must be one more (hidden) species having features A and H.

A similiar, but slightly more complicated, characterization of so-called interval multigraphs— intersection multigraphs of subpaths of some paths— has also been given by McKee.

Problem 17 *Is there some multigraph variant of* Bernstein *and* Goodman*'s Theorem? How could one decide whether a given chordal multigraph is representable on some given tree?*

Whereas it is still not known how th recognize k-line graphs, the multigraph version, intersection multigraphs of 3-uniform 3-conformal hypergraphs, can be recognized in polynomial time, by the standard approach of first determining all star graphs (with respect to 2-intersection) and then putting pieces together [P*]. Still, without 3-conformality, it is unknown.

16 Intersection bigraphs and digraphs

In intersection bigraphs, the sets come in two flavors, red and blue sets, and each set only recognizes intersection with sets of the different color. Therefore, the resulting intersection bigraph is bipartite. More formally, the intersection bigraph of a pair $((A, S_x/x \in U), (A, T_y/y \in W))$ of hypergraphs has $((A, S_x/x \in U), (A, T_y/y \in W))$ of hypergraphs has $V = U \cup W$ as vertex set, and an edge between $x \in U$ and $y \in W$ whenever $S_x \cap T_y \neq \emptyset$. Let us modify Example 6:

Example 9 *Now assume we no longer have mathematicians but members of a dancing club which only remember people of the other sex. Each member claims that she or he had a connected attendance time. Is the data of Figure 18 compatible with the model? Who, if any, is lying?*

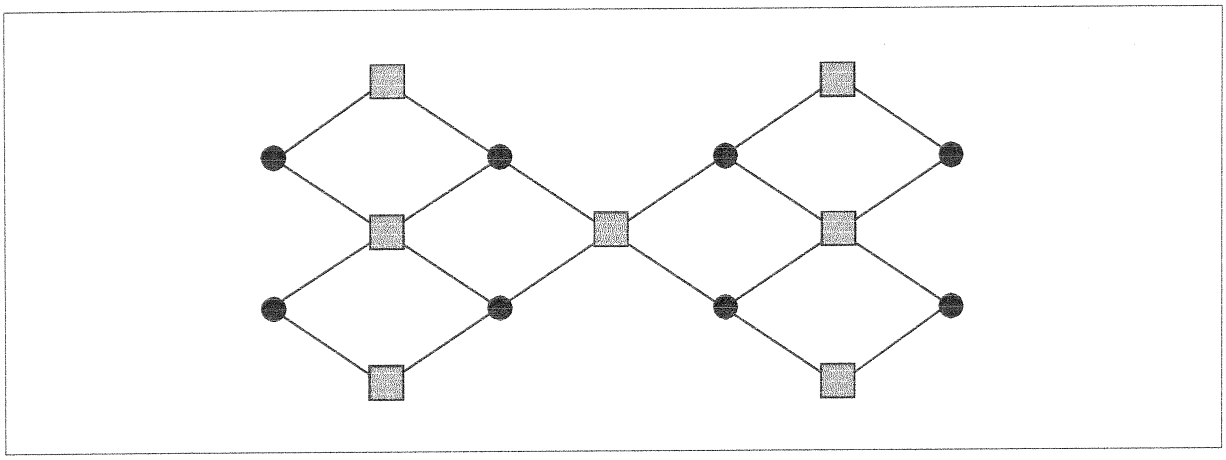

Figure 18: Who saw whom?

Intersection bigraphs contain less information than the corresponding intersection graphs. As for intersection multigraphs, the question whether this makes them easier or more difficult to recognize cannot be answered easily. It depends on the model. In some cases, the greater freedom enables us to represent *every* bigraph, for instance, it is easy to see that every bipartite graph is the intersection bigraph of subtrees of a tree, or convex sets in the plane, for instance.

There is only the following weak connection between intersection graphs and bigraphs: Let the bigraph $B(G)$ of a graph $G = (V, E)$ have $V \cup \{x'/x \in V\}$ as vertex set, and draw an edge between $x \in V$ and $y', y \in V$, whenever $xy \in E$ or $x = y$.

Proposition 16.1 *If G is intersection graph of $\mathcal{P}$-sets, then $B(G)$ is intersection bigraph of sets that are union of two intersecting $\mathcal{P}$-sets.*

The converse is in general false. For instance, $K_{3,3}$ is no line graph, but $B(K_{3,3})$ is a line bigraph (i.e. intersection bigraph of two families of 2-element sets).

Note also that $B(\overline{tK_2})$ is the so-called 'cocktail-party graph' $CP(t)$, obtained from $K_{t,t}$ by deleting a complete matching, which plays about the same role for bigraphs than the octahedrons for graphs.

16.1 Intersection digraphs

The *intersection digraph* of a family $(S_x, T_x), x \in V$ of *pairs* of sets has V as vertex set, and an arc from x to y whenever $S_x \cap T_y \neq \emptyset$.

This concept is older and more well-known than the concept of interval bigraphs, however, interval bigraphs are more general[43] The bigraph $B(D)$ of a digraph $D = (V, A)$ has $V \cup \{x'/x \in V\}$ as vertex set. There is an edge between $x \in V$ and $y', y \in V$ whenever $xy \in A$.

[43] except in cases where there is some additional restriction between the S-sets and the T-sets belonging to the same index, as, for instance for interval nest digraphs or interval catch digraphs, compare [E. Prisner, Algorithms for interval catch digraphs, Discrete Appl. Math. 51 (1994) 147-157].

Proposition 16.2 *[M97] D is intersection digraph of pairs of $\mathcal{P}$-sets if and only if $B(D)$ is intersection bigraph of $\mathcal{P}$-sets.*

16.2 Intersection bigraphs

For every intersection bigraph, we get again star graphs consisting of all vertices whose representative set covers some fixed single point. Surely the star graphs are complete bipartite.

16.2.1 Interval bigraphs

Interval digraphs have been introduced in [SDRW89]—we deal here with the more general bigraph model. There is a notion similiar to that of asteroidal triples in graphs: An *asteroidal triple of edges* is formed by three edges where any two are connected by a path avoiding vertices and neighbors[44] of the third edge.

Proposition 16.3 *[M97] No interval bigraph contains an asteroidal triple of edges*[45] *.*

Proof: Assume an interval bigraph contains an asteroidal triple $u(1)w(1), u(2)w(2), u(3)w(3)$ of edges. Then $S_{u(i)} \cap T_{w(i)}, i = 1,2,3$, are three disjoint intervals in $\mathbf{R}$. Assume $S_{u(i)} \cap T_{w(i)}$ lies between the other two. But then, for every path from $u(1)w(1)$ to $u(2)w(2)$, some representator must intersect $S_{u(2)} \cap T_{w(2)}$, a contradiction. QED

This analogue of asteroidal-triple-freeness of interval graphs surprisingly implies an analogue of chordality of interval graphs:

Corollary 16.4 *Interval bigraphs do not contain induced cycles of length 6 or more.*

There follows that for this model the converse of Proposition 16.1 holds:

Proposition 16.5 *A graph G is an interval graph if and only if $B(G)$ is an interval bigraph.*

Proof: Assume G is no interval graph. If G contains an asteroidal triple x, y, z, then we get an asteroidal triple xx', yy', zz' of edges in $B(G)$. Thus $B(G)$ is no interval bigraph. Otherwise, by Theorem 5.4, G must contain some induced C_ℓ with $\ell \geq 4$. But then $B(G)$ contains some induced C_m with $m \geq 6$, and again $B(G)$ is no interval bigraph. QED

Therefore, recognizing line bigraphs is at least as difficult than recognizing line graphs. The difficulty of recognizing interval bigraphs may be due to the fact that, although intervals of the real line have the Helly-property, the *bicliques*, i.e. inclusion-maximal complete bipartite subgraphs, are no longer necessarily star graphs. All one can show that for every biclique $U * W$, either $\bigcap_{u \in U} S_u$ or $\bigcap_{w \in W} T_w$ must be nonempty. Thus finding out the star graphs seems to be a lot more difficult in this model than for intersection *graphs*. The only known efficient method for recognizing interval bigraphs doesn't use star graphs but separators.

Theorem 16.6 *[M97] Interval bigraphs can be recognized in time $O(nm^6(n+m)\log n)$, where n and m are number of vertices and edges.*

Problem 18 *Is there some quicker or simpler recognition algorithm for interval bigraphs? Is there one which works by revealing the star graphs?*

For *proper interval bigraphs* we require that all intervalls S_x are pairwise incomparable, as well as all T_y. Proper interval bigraphs have a nice characterization which again uses the order of intervals on $\mathbf{R}$:

[44] where a neighbor of an edge is adjacent to at least one of its vertices

[45] One way of attacking Example 9 consists in searching for asteroidal triples of edges in Figure 18.

Theorem 16.7 *[S96] A bipartite graph is a proper interval bigraph if and only if it is a permutation graph.*

Proof: Let B be the intersection bigraph of the two families of intervals $([a_x, b_x]/x \in U)$ and $([c_y, d_y]/y \in W)$. We construct lines between the x-axis and the line $y = 1$ by defining R_x to be the line from $(a_x, 0)$ to $(b_x, 1)$, for $x \in U$ and otherwise, if $x \in W$, let R_x be the line from $(c_y, 1)$ to $(d_y, 0)$. Since $a_v \leq a_z < b_z \leq b_v$ is impossible for $v \neq z \in U$, R_v and R_z are disjoint, and the same if both v and z lie in W. On the other hand, for $x \in U$ and $y \in W$, $R_x \cap R_y \neq \emptyset$ if and only if $S_x \cap T_y \neq \emptyset$.

Assume conversely that the bipartite graph B with bipartition $U \cup W$ is a permutation graph, i.e. the intersection graph of a family $(R_z/z \in U \cup W)$ of straight lines between two parallel lines. It is not too difficult to show that this representation can be changed in such a way that for every $x \in W$, the lower endpoint sits to the left of the upper endpoint, and conversely for every $y \in W$. The remainder of the proof, the construction of the intervals, reverses the construction in the first part of the proof. QED

Therefore, proper interval bigraphs can be recognized in time $O(n^2)$.

16.2.2 Line bigraphs

The *line digraph* $L(D)$ of a digraph D is intersection digraph of the arc set of D, where an arc is viewed as a pair of vertices. Line digraphs are generally considered as the natural digraph analogue of line graphs. However, recognizing line digraphs is considerably easier than recognizing line graphs—if we do not have loops, we simply have to find all so-called dicliques[46], and essentially[47] compute its intersection digraph to find D. There is no ambiguity, as for line graphs, where we have to find out which triangles are star graphs. If we look closer on the concept, line digraphs are not the analogue of line graphs—instead, they are digraph analogues of unions of complete graphs. Interval digraphs of multigraphs are just intersection digraphs of families of pairs (S_x, T_x), where each S_x and each T_x contains exactly one element. Line graphs of multigraphs are intersection graphs of 2-element sets. Intersection graphs of 1-element sets are just unions of complete graphs.

Intersection bigraphs of pairs of 1-element sets are just unions of complete bipartite graphs. In other words, a digraph D is the line digraph of some multigraph iff ist bigraph $B(D)$ is the disjoint union of complete bipartite graphs.

What about discrete intersection bigraph models where the sets involved are larger? Intersection bigraphs of families $((S_x, T_x), x \in V)$, where the S_x are 1-element sets and the T_x are k-element sets, for fixed k, are almost as easy to recognize, as is left to the reader.

Thus the first challenge, and the real bigraph generalization of line graphs, would be the case where all S_x and all T_x contain 2 elements. We could view this as intersection bigraphs B of pairs G_1, G_2 of graphs with the same vertex set.

Obviously all star graphs are complete bipartite, but not all of them are bicliques. On the other hand, there are several other types of bicliques in B, as indicated in Figure 19.

It turns out that the recognition problem is easy if we require large enough minimum degree on both G_1 and G_2.

Under the condition $\delta(G_1), \delta(G_2) \geq 4$, all star graphs are bicliques, and more important, all bicliques are star graphs. Then recognition is straightforward.

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[46] ordered variants of bicliques

[47] There are, however, some technical details: We have to check whether every vertex lies in at most one head of a biclique and at most one tail of a biclique. If some vertex does not lie in some head, then we have to add a artificial pair $(\emptyset, \{v\})$ which gets a source in D, and the same for sinks. We may also sample some or all vertices $v_1, v_2, \ldots, v_s$ which lie in no head of a diclique, and instead of adding all these separate new pairs, we add $(\emptyset, \{v_1, v_2, \ldots, v_s\})$

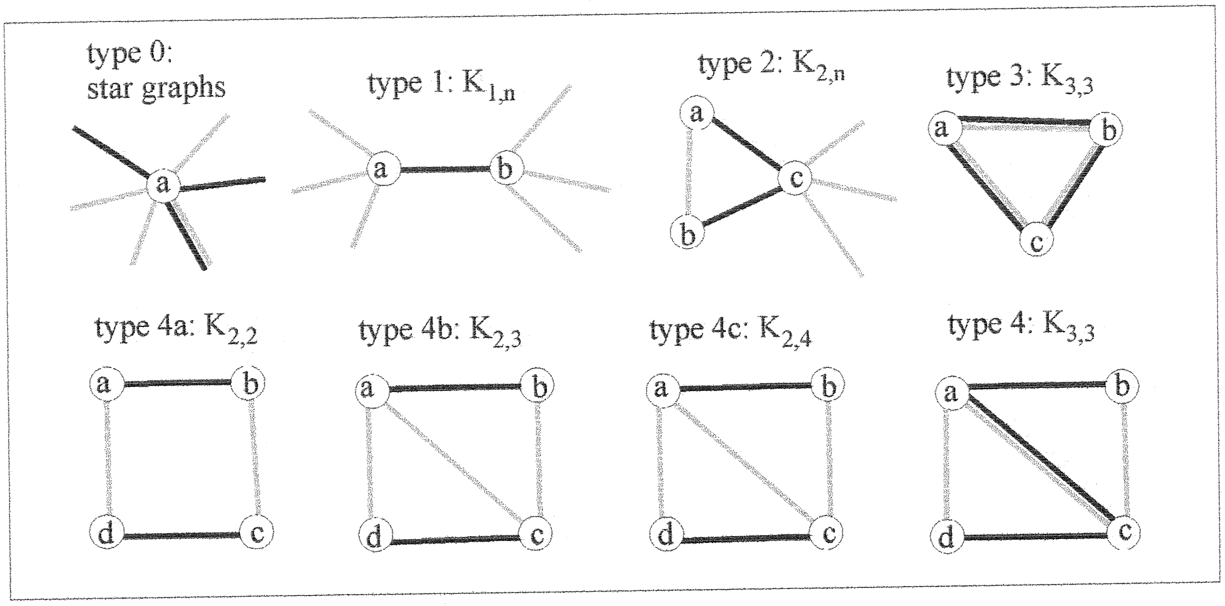

Figure 19: Types of bicliques in line bigraphs.

Let us now relax to the condition $\delta(G_1), \delta(G_2) \geq 3$. Again all star graphs are bicliques, and there are only two further types of bicliques, type 3 and type 4 in Figure 19. Note that every vertex of B lies in exactly two star graphs, and every edge in one or two of them. Our strategy is now as follows: First we compute all large bicliques[48], where a biclique is called *large* if it contains $K_{3,3}$. We compute the intersection multigraph M of the large bicliques. All large bicliques are originally unlabeled. Now we label essentially all vertices of M by '*', '3', or '4', depending on whether the biclique is a star graph or has type 3 or 4 in Figure 19. Starting with the assumption that one particular large biclique B_0 is a star graph[49], we proceed to label the vertices of M by five rules. The simplest of them says, that if a vertex labeled by '*' is connected to another vertex by a *single* edge in M, then this other vertex also gets the label '*'.

If B is indeed a line bigraph with G_1, G_2 as above, and if our assumption on B_0 being a star graph was correct, then we succeed to label almost all vertices of M correctly. There remain some 'white spots', corresponding to K_4s in $G_1 \cap G_2$, which however do not harm.

Theorem 16.8 *[P***] There is a polynomial-time algorithm that decides for any bipartite graph B whether it is the line bigraph of a pair (G_1, G_2) of graphs on the same vertex set, with $\delta(G_1), \delta(G_2) \geq 3$.*

Problem 19 *What is the complexity of recognizing general line bigraphs?*

17 Connection Graphs

Connection graphs are essentially underlying graphs of intersection digraphs. The only difference is that loops of the digraph remain in the connection graph, i.e. connection graphs are undirected graphs with loops allowed. Formally, we have a family $(S_x, T_x), x \in V$ of *pairs* of sets. The *connection graph* has V as vertex set, and an edge between x to y whenever $S_x \cap T_y \cup S_y \cap T_y \neq \emptyset$ [M95].

Connection graphs extend intersection graphs—every intersection graph of a family of $\mathcal{P}$-sets is a connection graph of a family of pairs of $\mathcal{P}$-sets after adding all loops. Therefore recognition for a class of connection graphs may be more difficult than for the corresponding intersection class. For instance, if the property $\mathcal{P}$ is to have just one element, then intersection bigraphs,

[48] which can be done in this special case rather easily, since each $K_{3,3}$ lies in a unique biclique

[49] If B contains bicliques containing $K_{3,4}$, we choose this as B_0.

digraphs and graphs are fairly easy to recognize, but recognizing the corresponding connection graphs is NP-complete [CE90]. However, under some minimum degree condition, recognition is again possible essentially by the (bi)clique multigraph method [P97].

Why bicliques? Star graphs for connection graphs still have a slightly more general shape. If we consider any point $a \in \bigcup_{x \in V}(S_x \cup T_x)$, then we have the set V_1 of all $x \in V$ where $a \in S_x$, and the set V_2 of all $y \in V$ where $a \in T_y$. Every $x \in V_1$ is adjacent to every $y \in V_2$. Note also that V_1 and V_2 are not necessarily disjoint. Certainly, all $x \in V_1 \cap V_2$ have loops. Finally these subgraphs are not necessarily induced in G, nor are they necessarily maximal.

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