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Radical Equations

Teacher: Hello, welcome to the mini-lesson on how to solve radical equations. By the way, a **radical equation** is any equation containing radicals.

Teacher: Does anybody remember our general strategy for solving equations?

Student: Don't we try to simplify it?

Teacher: Sure. We try to simplify both expressions on the sides of the equation, but we also try to simplify the equation. Does anybody remember how? What can we do to an equation to simplify it?

Student: We can add or subtract the same on both sides. I guess we can also multiply the same on both sides.

Teacher: Correct, although remember that when multiplying you have to be careful. If you multiply both sides by 0 you get an equation having more solutions than the equation before.

Student: But why would we multiply both sides by 0? That sounds such a silly thing to do.

Clever Student: Yes, but we may not even notice that we multiply by 0. Imagine we multiply an equation by $x-1$ at some point, and later we find that $x=1$ is a solution of the resulting equation. That means that if we look at $x=1$, we had multiplied the equation by $x-1=0$. Therefore 1 may not be a solution of the original equation.

Teacher: Right, and in such a case we have to check explicitely whether $x=1$ is a solution. If not, we call it a *fake solution.* A solution to the transformed equation, but not of the original one.

Teacher: But let's get back to the allowed operations: Adding subtracting, or multiplying the same (not equal to 0) on both sides of the equation is allowed. What about division?

Clever Student: Dividing means multiplying by its reciprocal, so yes, this is allowed too.

Teacher: Right, as long as we don't try to divide by 0.

Teacher: So what about the equation

$$\sqrt{x-1}=2?$$

How would we simplify it? Would adding, subtracting, multiplying, dividing help?

Student: Couldn't we simplify $\sqrt{x-1}$ first?

Clever Student: No, that's not possible. Although square roots of products and quotients can be simplified, there is no formula for the square root of a sum or of a difference. But couldn't we just square both sides of the equation? This way the radical would go.

Teacher: Yes, that's it. The square of a square root of something is just this something. So on the left side we just get $x-1$. On the right side we get the square of 2, which equals 4. Overall we get

$$x-1=4$$

which can be easily finished—we get $x = 5$. Plugging this number into our intial equation shows us that 5 is indeed a solution, the square root of $5 - 1$, or of 4, is indeed equal to 2. $\sqrt{5-1} = 2$.

Teacher: Therefore squaring both sides of an equation seems to be a valid change of the equation too, one that replaces an equation by an *equivalent* equation—meaning by one with exactly the same solutions. But it is not, at least not always. The reason is that the squaring function is not a 1-1 function. After squaring two expressions or numbers, left and right might be equal, even if the intial ones were not. The squares of 2 and -2 are equal, but 2 is not equal to -2.

Teacher: Look at the example

$$\sqrt{x-1} = -2.$$

Student: Let me try: Squaring both sides we get $x - 1 = 4$ or $x = 5$. The same solution as the previous example?

Clever Student: Not really. $x = 5$ is **not** a solution of the equation $\sqrt{x-1} = -2$, since the square root of $5 - 1$ equals 2 and not -2. So we have a **fake solution** here, right?

Teacher: Right. Whenever you square both sides of an equation you always have to check whether the solution you eventually get of the repeatedly transformed equation is also a solution of the initial equation. If not, we call it a **fake solution**.

Teacher: Let's try another, a more complicated example. What about

$$\sqrt{2x^2-1} + x = 0$$

Student: We square?

Teacher: Why would we square? What would we get?

Student: Since the square of a square root of something is just something—wouldn't we get $2x^2 - 1 + x^2 = 0$?

Clever Student: Nonsense! On the left you have a sum of a radical and the expression x. The square of a sum is **not** equal to the sum of the squares. Rather you would have to FOIL the expression on the left, and you would get

$$(2x^2-1) + x\sqrt{2x^2-1} + \sqrt{2x^2-1}x + x^2 = 0,$$

which can be simplified to

$$(2x^2-1) + 2x\sqrt{2x^2-1} + x^2 = 0.$$

Student: But this equation looks even more complicated than the intial one!

Teacher: And it is. That's why we should not square the intial equation, at least not yet. First we isolate the radical on one side. This way, when squaring we don't square a sum, but rather a square root, and really get the "something".

Clever Student: So, to isolate the radical, we first subtract x on both sides and get

$$\sqrt{2x^2-1} = -x?$$

Teacher: Right.

Student: But **now** we can square, right? We get

$$2x^2 - 1 = (-x)^2 = x^2,$$

right?

Teacher: Right.

Student: But this is a very simple quadratic equation! So simple that we don't even need quadratic formula. We subtract x^2 on both sides to get $x^2 - 1 = 0$, or $x^2 = 1$, or $x = \pm 1$. We get two solutions: 1 and -1.

Clever Student: Wait a second. Don't we have to check for fakeness? 1 does not seem to work, since $\sqrt{2 \cdot 1 - 1} + 1$ equals 2 but not 0. -1 however works. $\sqrt{2 \cdot 1 - 1} - 1$ equals 0. So 1 is a fake solution and -1 is really a solution, the only one.

Teacher: So here is the procedure for solving a radical equation:

1. Isolate the radical
2. Square the equation (or raise to the power of 3, 4, depending on the radical)
3. Solve the resulting (hopefuly simpler) equation
4. Check for fakeness.

Clever Student: What if we have an equation with two radicals, like in

$$\sqrt{3x} - \sqrt{2x - 2} - 1 = 0.$$

Teacher: This can be done, but only if we have two square roots. In that case we cannot isolate both radicals. So, after squaring, only the isolated radical will go. On the other side, we have the other radical and some more expression—if we square this, we have to FOIL, and the radical will still be there. But in the transformed equation, we have only one radical left. To this equation we can apply the procedure described above.

Thus in case of two radicals, there are two more steps that have to be performed first:

-1. Isolate one of the radicals

0. Square the equation and simplify

1. Solve the resulting equation with one radical.

and of course—the last step is done using the four-step procedure for equations with just one square root discussed above.

Clever Student: So let's isloate the $\sqrt{3x}$. We get

$$\sqrt{3x} = \sqrt{2x - 2} + 1$$

We square both sides, FOILING the right, and get

$$3x = (2x - 2) + 2\sqrt{2x - 2} + 1$$

Now we isolate the other radical and get

$$x + 1 = 2\sqrt{2x - 2}.$$

Teacher: Actualy the radical is not entirel isolated, but the factor of 2 does not hurt when squaring.

Clever Student: Right, so we square, FOIL this time on the left, and get

$$x^2 + 2x + 1 = 4(2x - 2)$$

or the quadratic equation $x^2 - 6x + 9 = 0$, which has a double solution $x = 3$

Student: Which checks. $\sqrt{3 \cdot 3} - \sqrt{2 \cdot 3 - 2} - 1 = 0$.

Student: Now with equations with three radicals, as

$$\sqrt{x + 1} = \sqrt{2x - 1} + \sqrt{3x + 2} + 2$$

it is probably the same, just with one more isolating-squaring loop, right?

Teacher: Unfortunately no. After isolating one radical and squaring the number of radicals is not reduced to 2. So usually we cannot do three-radical equations.

But so far we only looked at square roots. What we also can do is radical equations with cube roots or other roots, as long as we have only one of them. We just don't isolate and square but rather isolate and cube.