MAT200
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Antiderivatives

Hello, welcome to the Calculus mini-lecture on antiderivatives.

We all know by now what derivatives are, what differentiating a function means. But what are **antiderivatives**?

Antidifferentiating a function means reverting the process of differentiating a function. Let me explain this by an analogy:

Assume you are back in middle school and have just learnt how to square any number. So, typical questions are: Given a number, like 3, what is its square? Now imagine that suddenly the teacher starts to ask different kinds of questions. The teacher gives you the result of the squaring process—the square of a number—, and asks what the initial number—the number that has been squared—has been. So a typical question now would be: The square of a number is 9. Which number has been squared? Well, obviously in this example it was 3 or -3. These numbers whose square equals 9 are called the **square roots** of 9.

Back at College, you have just learnt how to differentiate any given function. For instance, the derivative of $f(x) = \frac{1}{4}x^4$ is $f'(x) = x^3$. Now suddenly the teacher starts asking different kinds of questions, as: The derivative of an unknown function f is $f'(x) = x^3$. What was the initial function f? Any such possible function is called an **antiderivative** of x^3.

Thus an **antiderivative** of a given function g is any function whose derivative equals g. f is an antiderivative of g if g is the derivative of f.

In our example, an antiderivative of the function $g(x) = x^3$ is, for example, the function $f(x) = \frac{1}{4}x^4$. But it is obviously not the only one. Another antiderivative of the function x^3 is the function defined by $\frac{1}{4}x^4 + 1$, another one is defined by the expression $\frac{1}{4}x^4 + 2$, and so on. Therefore functions usually have more than one antiderivative, similar to numbers having usually more than one square root.

If a function F is an antiderivative of the function f, then any function obtained from F by adding or subtracting any number is also an antiderivative of f. The graphs of all these functions are obtained from the graph of F by shifting it up or down. It can also be shown that this way we really get **all** antiderivatives of f—there are no others.

Now we need some notation. This symbol here, this fancy capital S, together with $f(x)$ and dx indicates the set of **all** antiderivatives of the function f. Note the three parts: First the S, telling that we look for antiderivatives, then the function for which we want to find the antiderivative, and finally the dx tells us that x is supposed to be the variable (in case the expressen contains more than one letter, say x and a.) This dx part looks a little silly at the moment, but will become clearer later.

So this symbol above stands for one fixed antiderivative F, together with all functions one obtains from F by adding (or subtracting) any number. Instead of listing all these infinitely many antiderivatives, for which we obviously don't have time, we usually just write down one of these antiderivatives, as F, and write down $+C$. C stands for a constant and indicates that we can add (or subtract) any number.

In our example, the set of all antiderivatives of the function defined by x^3 is $\frac{1}{4}x^4 + C$, meaning that they are all functions obtained from the function $F(x) = \frac{1}{4}x^4$ by adding or subtracting any number.

The set of all antiderivatives of the function f is also called the **indefinite integral** of f.

OK, so now we have a problem and we have definitions, but how can we actually **find** these antiderivatives? Well, there are some simple, elementary rules for antidifferentiation, which follow directly from the elementary rules for differentiation. This is again quite similar to the squaring—square root example. Since we know that the square of a product equals the product of the squares of the factors, there follows that the square root of a product equals the the product of the square roots of the factors.

Back to Calculus, we know that the derivative of a sum of functions equals the sum of the derivatives, therefore the antiderivative of a sum of functions must be equal to the sum of the antiderivatives of the parts. The hope, of course, is that the antiderivatives of the summands are easier to find than the antiderivative of the whole sum, it's a so-called "divide and conquer" strategy. In the same way the antiderivative of a difference of functions equals the difference of the antiderivatives of the functions. Moreover the antiderivative of a multiple of a function equals the multiple of the antiderivative of the function.

Let's now discuss special functions: We know that the derivative of the exponential function e^x is the same function e^x, therefore e^x is also its own antiderivative.

For power functions we know that their derivatives are multiples of other power functions, the derivative of x^n is nx^{n-1}. Therefore we can also find antiderivatives of power functions. We push the constants a little back and forth and obtain that the antiderivative of x^n is $\frac{1}{n+1}x^{n+1} + C$. Instead of decreasing the exponent, which we do when differentiating, we increase the exponent when antidifferentiating, and instead of multiplying by the old exponent, we now divide by the new exponent $n + 1$.

There is one exception. Note that the rule works only for $n \neq -1$, since for $n = -1$ we would have to divide by $n + 1 = 0$, which is impossible. But we know that the derivative of $\ln(x)$ is $\frac{1}{x}$, thus the antiderivative of $f(x) = \frac{1}{x} = x^{-1}$ is $\ln(x)+C$, which settles this one exception. Actually we should be a little more precise here and say that the antiderivatives of $f(x) = \frac{1}{x}$ are $\ln|x| + C$. While $\ln(x)$ is only defined for positive values x, $\ln|x|$ extends this function to all numbers $\neq 0$, and is therefore a more appropriate antiderivative of $\frac{1}{x}$ with the same domain.

So we have formulas for antiderivatives of the natural exponential function and of all power functions, but at the moment we don't know the antiderivatives of $\ln(x)$ yet. We will obtain this later, using an advanced technique called **integration by parts**.

Let me close with a warning: We don't have a formula for antiderivatives of products, or quotients, or composed functions, and there are no such formulas working in general.

It get's even worse: While we can differentiate every function, for most functions, obtaining the formulas for antiderivatives is difficult, often even impossible. Antidifferentiating is often more an art than a science.

Thanks for your patience.